Groups of isometries associated with automorphisms of the half - plane

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The study of integral operators on spaces of analytic functions has been considered for the past few decades. However, most of the studies in this line are based on spaces of analytic functions of the unit disc. For the analytic spaces of the upper half-plane, the literature is still scanty. Most notable is the recent work of Siskakis and Arvanitidis concerning the classical Cesàro operator on Hardy spaces of the upper half-plane. In this dissertation, we characterize all continuous one-parameter groups of automorphisms of the upper half-plane according to the nature and location of their fixed points into three distinct classes, namely, the scaling, the translation, and the rotation groups. We then introduce the associated groups of weighted composition operators on both Hardy and weighted Bergman spaces of the half-plane. Interestingly, it turns out that these groups of composition operators form three strongly continuous groups of isometries. A detailed analysis of each of these groups of isometries is carried out. Specifically, we determine the spectral properties of the generators of every group, and using both spectral and semigroup theory of

Banach spaces, we obtain concrete representations of the resolvents as integral operators on both Hardy and Bergman spaces of the half-plane. For the scaling group, the resulting resolvent operators are exactly the Cesàro-like operators. The spectral properties of the obtained integral operators is also determined. Finally, we detail the theory of both Szegö and Bergman projections of the half-plane, and use it to determine the duality properties of these spaces. Consequently, we obtain the adjoints of the resolvent operators on the reflexive Hardy and Bergman spaces of the half-plane.

Key words: Semigroups, Automorphism groups, Composition and Integral operators, Duality properties, Spectral properties, Resolvents, Infinitesimal generator, Strong continuity, Reflexive Banach spaces

DEDICATION

To my family.

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INTRODUCTION

1.1 History and Motivation

The use of semigroup theory approach in the study of integral operators on the spaces of analytic functions dates back to some decades ago. The most common example of integral operators is probably the classical Cesàro operator defined on analytic spaces of the unit disc, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, as

$$\mathbf{C}_1 f(z) = \frac{1}{z}\int_0^z \frac{f(\omega)}{\omega - 1}\, d\omega. \tag{1.1}$$

Siskakis [35] realized $\mathbf{C}_1$ as a resolvent at 0 of the infinitesimal generator of a certain semigroup of weighted composition operators on the Hardy space of the unit disc, $H^p(\mathbb{D})$, then determined its norm and spectrum on $H^p(\mathbb{D})$. In the same year, he used a specific semigroup to obtain information on the averaging operator, $\mathcal{A}$, defined by

$$\mathcal{A}f(z) = \frac{1}{z-1}\int_1^z f(\omega)\, d\omega, \tag{1.2}$$

and acting on the weighted Bergman spaces of the disc, $L^p_a(\mathbb{D}, m_\alpha)$, see [36]. Specifically, he established the boundedness and spectrum of $\mathcal{A}$ on $L^p_a(\mathbb{D}, m_\alpha)$. Siskakis later in [37] extended the results obtained for $\mathbf{C}_1$ on $H^p(\mathbb{D})$ to the weighted Bergman spaces, $L^p_a(\mathbb{D}, m_\alpha)$.

Earlier in 1984, Cowen [12] had given an alternative proof to the Kriete-Trutt theorem of the subnormality of $\mathbf{C}_1$ on $H^2(\mathbb{D})$ based on the connection between $\mathbf{C}_1$ and composition semigroups on $H^2(\mathbb{D})$. In [29], the semigroup theory even proved to be a more powerful method in the study of integral operators, especially the Cesàro operators. Using this technique in [29], Miller, Miller and Smith showed that, analogous to the Kriete-Trutt result, $\mathbf{C}_1$ is subdecomposable on $H^p(\mathbb{D})$, $1 \leq p < \infty$. The decomposability property of $\mathbf{C}_1$ was later extended to the case of unweighted Bergman spaces, $L^p_a(\mathbb{D}, m_0)$ by Miller and Miller [28].

A unified approach that resolved the problem mentioned in [25] as to whether $\mathbf{C}_1$ is subdecomposable on $H^1(\mathbb{D})$ or $L^1_a(\mathbb{D}, m_\alpha)$, was given by Persson in [32]. Her method combined some ideas of Dahlner [13] with a technique based on different semigroups of composition operators.

Recently, Ballamoole, Miller and Miller [7] extended the above properties to the Cesàro-like operator $\mathbf{C}_\nu$ acting on both $H^p(\mathbb{D})$ and $L^p_a(\mathbb{D}, m_\alpha)$ spaces, $1 \leq p < \infty$, and is defined by

$$\mathbf{C}_\nu f(z) = \frac{1}{z^\nu} \int_0^z \frac{f(\omega)\omega^{\nu-1}}{1-\omega}\, d\omega. \tag{1.3}$$

In another direction, Arvanitidis and Siskakis [4] considered the Cesàro operator $\mathcal{C}_1$ which is acting on spaces of analytic functions of the upper half-plane, $\mathbb{U} := \{z \in \mathbb{C} : \Im(z) > 0\}$, and is defined by

$$\mathcal{C}_1 f(z) = \frac{1}{z} \int_0^z f(\omega)\, d\omega. \tag{1.4}$$

Specifically, they realized $\mathcal{C}_1$ as a resolvent at point $1 - \frac{1}{p}$, $p > 1$, of the generator of a strongly continuous group on the Hardy spaces of the half-plane, $H^p(\mathbb{U})$, and determined its spectral properties. Moreover, in [26], Liu and Zhou determined norm bounds of a class of integral operators induced by weighted Bergman projections in the upper half-plane.

On the other hand, based on the above remarks, it is apparent that the literature on the study of integral operators on the spaces of analytic functions of the upper half-plane $\mathbb{U}$ is much less complete than in the setting of the unit disc $\mathbb{D}$. In the same line, the available studies on the upper half-plane $\mathbb{U}$ focus more on the composition operators as opposed to the integral operators. See for instance, [17, 27, 39] for some studies on composition operators on $H^p(\mathbb{U})$ and $L^p_a(\mathbb{U}, \mu_\alpha)$. For such reasons, the work by Arvanitidis and Siskakis [4] concerning the Cesàro operator $\mathcal{C}_1$ stands out, and motivates this study a great deal. It's therefore natural to generalize their results for the Hardy spaces $H^p(\mathbb{U})$; complete their analysis for the case $p = 1$, and even consider extensions to the setting of weighted Bergman spaces, $L^p_a(\mathbb{U}, \mu_\alpha)$.

We denote the Cesàro-like operators on spaces of analytic functions of the upper half-plane $\mathbb{U}$ by $\mathcal{C}_\nu$, and it is formally defined as,

$$\mathcal{C}_\nu f(z) = \frac{1}{z^\nu} \int_0^z f(\omega)\omega^{\nu-1}\, d\omega. \tag{1.5}$$

Just as in the case of Cesàro-like operators on the disc $\mathbb{D}$ (see [7]), these operators arise as resolvents.

The outline of this dissertation is as follows: Chapter 1 is introductory and we give some definitions and statements of fundamental results that will be of great use in the subsequent chapters. Chapter 2 is devoted to the basic properties of both Hardy and Bergman spaces of the upper half-plane. Apart from stating the well known facts about these spaces, we also prove some simple results. The theory of Bergman projections of the upper half-plane is well presented and used to prove the duality properties of these spaces.

In Chapter 3, we characterize the one-parameter groups of automorphisms of the upper half-plane into three types and introduce the corresponding groups of weighted composition operators acting on both Hardy and Bergman spaces of the half-plane. The three groups of weighted composition operators in turn form strongly continuous groups of invertible isometries, namely, the scaling, the translation, and the rotation groups.

It is in Chapter 4 and 5 where we present detailed analysis of each of the strongly continuous groups of isometries acting on Hardy and Bergman spaces of the half-plane. Using both semigroup and spectral theory, we obtain the norms, spectra, adjoints, resolvents and related properties of these groups. Some of these results have already been accepted for publication as indicated in [6]. Conclusions and future directions of this research are given in the last Chapter 6.

1.2 The unit disc and the upper half - plane

Let $\mathbb{C}$ be the complex plane. The set

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\},$$

is called the (open) unit disc. Let dA denote the area measure on $\mathbb{D}$, normalized so that the area of $\mathbb{D}$ is one. In terms of rectangular and polar coordinates, we have

$$dA(z) = \frac{1}{\pi}\, dx\, dy = \frac{r}{\pi}\, dr\, d\theta,$$

where $z = x + iy = re^{i\theta} \in \mathbb{D}$. For $\alpha \in \mathbb{R}$, $\alpha > -1$, we define a positive Borel measure dm_α on $\mathbb{D}$ by

$$dm_\alpha = (1 - |z|^2)^\alpha dA(z).$$

It is clear that dm_α is a finite measure on $\mathbb{D}$. In fact if $\alpha = 0$, then dm_0 and dA coincide and therefore we shall simply denote it by dA. Thus we consider dm_α as a weighted measure and a generalization of dA.

On the other hand, the set

$$\mathbb{U} = \{\omega \in \mathbb{C} : \Im(\omega) > 0\}$$

denotes the upper half of the complex plane $\mathbb{C}$. For $\alpha > -1$, we define a weighted measure on $\mathbb{U}$ by

$$d\mu_\alpha(\omega) = (\Im(\omega))^\alpha \, dA(\omega),$$

where $\omega \in \mathbb{U}$. Again, it can be easily seen that $\alpha = 0$ coincides with the unweighted Lebesgue measure.

The function

$$\psi(z) := \frac{i(1+z)}{1-z}$$

is referred to as the Cayley transform and maps the unit disc $\mathbb{D}$ conformally onto the upper half-plane $\mathbb{U}$ with inverse

$$\psi^{-1}(\omega) = \frac{\omega - i}{\omega + i}.$$

1.3 Spectra of linear operators

Let $(X, \|.\|)$ and $(Y, \|.\|)$ be Banach spaces over $\mathbb{C}$. The space

$$\mathcal{L}(X,Y) = \{T : X \to Y : T \text{ is linear and continuous}\}$$

endowed with the operator norm $\|T\| = \sup_{\|x\| \leq 1} \|Tx\|$ is in turn a Banach space and we abbreviate $\mathcal{L}(X) := \mathcal{L}(X,X)$. More generally, a linear operator from X into Y is a linear map A defined on a linear subspace $\mathcal{D}(A)$, that is, $A : \mathcal{D}(A) \subset X \to Y$. $\mathcal{D}(A)$ is called the domain of A.

We define the kernel and respectively, the range of A by

$$\ker(A) := \{x \in \mathcal{D}(A) : Ax = 0\},$$

and

$$\mathcal{R}(A) := \{y \in Y : \exists\, x \ \in \mathcal{D}(A) \ \text{with y = Ax } \}.$$

A is said to be a closed operator if its graph, $\{(x, Ax) : x \in \mathcal{D}(A)\}$, is closed in $X \times Y$. The closed graph theorem asserts that if $\mathcal{D}(A) = X$, then A is continuous if and only if A is closed.

1.3.1 The spectrum

Let A be a closed operator on X. The resolvent set of A, $\rho(A)$, is given by

$$\rho(A) = \{\lambda \in \mathbb{C} : \lambda I - A : \mathcal{D}(A) \to X \text{ is bijective}\},$$

and its spectrum, $\sigma(A)$, by

$$\sigma(A) = \mathbb{C} \setminus \rho(A).$$

The spectral radius of a bounded operator $A \in \mathcal{L}(X)$ is defined by $r(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\}$, with the relation that $r(A) \leq \|A\|$. The point spectrum of A, $\sigma_p(A)$, is defined by

$$\sigma_p(A) = \{\lambda \in \mathbb{C} : \text{ there exists some } x \in \mathcal{D}(A) \setminus \{0\} \text{ with } \lambda x = Ax\} \subseteq \sigma(A),$$

where we call $\lambda \in \sigma_p(A)$ an eigenvalue of A and the corresponding v an eigenvector of eigenfunction of A. Further, the sets

$$\sigma_{ap}(A) = \{\lambda \in \mathbb{C} : \lambda I - A \text{ is not injective or } \mathcal{R}(\lambda I - A) \text{ is not closed in } X\},$$

$$\sigma_{su}(A) = \{\lambda \in \mathbb{C} : \lambda I - A \text{ is not surjective }\}, \quad \text{and}$$

$$\sigma_r(A) = \{\lambda \in \mathbb{C} : \mathcal{R}(\lambda I - A) \text{ is not dense in } X\},$$

are called the approximate point spectrum, the surjectivity spectrum, and the residual/ compression spectrum of A, respectively. For $\lambda \in \rho(A)$ the operator

$$R(\lambda, A) := (\lambda I - A)^{-1} : X \to X$$

is, by the closed graph theorem, a bounded operator on X and is called the resolvent (of A at the point λ), or simply the resolvent operator. In fact, $\rho(A)$ is an open subset of $\mathbb{C}$, and $R(\cdot, A) : \rho(A) \to \mathcal{L}(X)$ is analytic, see [14, Chapter VII].

Theorem 1 (Spectral Mapping Theorem for Resolvents)

Let A a be closed operator on X and $\lambda \in \rho(A)$. Then the following assertions hold;

1. $\sigma(R(\lambda, A))\backslash\{0\} = (\lambda - \sigma(A))^{-1} = \left\{\frac{1}{\lambda-\mu} : \mu \in \sigma(A)\right\}.$
2. $\sigma_j(R(\lambda, A))\backslash\{0\} = (\lambda - \sigma_j(A))^{-1}$ *for* $j = p,\ ap,\ su,\ r.$
3. *If x is eigenvector for the eigenvalue $\mu \neq 0$ of $R(\lambda, A)$, then $y = \mu R(\lambda, A)x$ is an eigenvector for the eigenvalue $\nu = \lambda - \frac{1}{\mu}$ of A. If $y \in \mathcal{D}(A)$ is an eigenvector for the eigenvalue $\nu = \lambda - \frac{1}{\mu}$ of A with $\mu \in \mathbb{C}\backslash\{0\}$, then $x = \mu^{-1}(\lambda y - Ay)$ is an eigenvector for the eigenvalue μ of $R(\lambda, A)$.*
4. $r(R(\lambda, A)) = \frac{1}{\operatorname{dist}(\lambda, \sigma(A))}.$
5. *If A is unbounded (that is, $\mathcal{D}(A) \neq X$), then $0 \in \sigma(R(\lambda, A))$.*

For a comprehensive account of theory of the spectra, we refer to [11], [18], and [34].

1.3.2 Adjoint of a linear operator

Let A be a linear operator from X to Y with dense domain. We define its adjoint A^* from Y^* to X^* by setting

$$\mathcal{D}(A^*) = \{y^* \in Y^* : \exists z^* \in X^* \text{ s.t. for all } x \in \mathcal{D}(A), \langle Ax, y^* \rangle = \langle x, z^* \rangle\},$$

and define $A^*y^* = z^*$. Since $\mathcal{D}(A)$ is dense, there is at most one vector $z^* = A^*y^*$ (as above) so that $A^* : \mathcal{D}(A^*) \to X^*$ is a function. It's clear that A^* is linear and closed from Y^* to X^*. The following result characterizes the relation between spectrum of an operator and that of its adjoint.

Proposition 1

Let A be a closed operator on X with dense domain. Then the following assertions hold;

1. $\sigma(A) = \sigma(A^*)$ *and* $R(\lambda, A)^* = R(\lambda, A^*)$ *for every* $\lambda \in \rho(A)$,
2. $\sigma_r(A) = \sigma_p(A^*)$, *while* $\sigma_p(A) \subseteq \sigma_r(A^*)$,
3. $\sigma_{ap}(A) = \sigma_{su}(A^*)$, *and* $\sigma_{su}(A) = \sigma_{ap}(A^*)$.

1.4 Semigroup theory of linear operators

In this section, we give a brief exposition of the theory of semigroups of linear operators on Banach spaces. We refer to [14], [18], or [30] for a detailed exposition of the theory. Let X be a Banach space. A one-parameter family $(T_t)_{t\geq 0} \subset \mathcal{L}(X)$ is a semigroup of bounded linear operators on X if

(i) $T_0 = I$, (I is the identity operator on X),

(ii) $T_{t+s} = T_t T_s$ for every $t,\ s\ \geq 0$ (the semigroup property).

1.4.1 Strongly continuous semigroup

A semigroup $(T_t)_{t\geq 0}$ of bounded linear operators on X is strongly continuous if

$$\lim_{t\to 0^+} T_t x = x \text{ or } \lim_{t\to 0^+} \|T_t x - x\| = 0 \text{ for every } x \in X.$$

Equivalently by the uniform boundedness principle, $(T_t)_{t\geq 0}$ is strongly continuous if $\forall\, t \in [0,\infty)$ and $x \in X$, $\|T_s x - T_t x\| \to 0$ as $s \to t$ in $[0,\infty)$. See [22, Chapter XII]. The infinitesimal generator, Γ, of the semigroup $(T_t)_{t\geq 0}$ is defined by

$$\Gamma x := \lim_{t\to 0^+} \frac{T_t x - x}{t} = \frac{\partial}{\partial t}(T_t x)\bigg|_{t=0} \tag{1.6}$$

for each $x \in \mathcal{D}(\Gamma)$, where the domain of Γ is given by

$$\mathcal{D}(\Gamma) = \left\{ x \in X : \lim_{t\to 0^+} \frac{T_t x - x}{t} \text{ exists} \right\}. \tag{1.7}$$

The next proposition summarizes the basic properties of an infinitesimal generator of a strongly continuous semigroup.

Proposition 2 ([18, Theorem 1.4])

The generator of a strongly continuous semigroup is a closed and densely defined linear operator that determines the semigroup uniquely.

One of the most powerful characterization of infinitesimal generators of strongly continuous semigroups is given by the Hille - Yosida theorem below. Specifically, it gives the conditions on the behavior of the resolvent of Γ which are necessary and sufficient for Γ to be the infinitesimal generator of strongly continuous semigroup, see [30].

Theorem 2 (Hille - Yosida Theorem)

A linear operator Γ is the infinitesimal generator of a strongly continuous semigroup of contractions $(T_t)_{t\geq 0}$ if and only if

(i) Γ *is closed and* $\overline{\mathcal{D}(\Gamma)} = X$,

(ii) *The resolvent set* $\rho(\Gamma)$ *of* Γ *contains* $\mathbb{R}^+$ *and for every* $\lambda \geq 0$,

$$\|R(\lambda, \Gamma)\| \leq \frac{1}{\lambda}. \tag{1.8}$$

In that case, if $h \in X$, *then*

$$R(\lambda, \Gamma)h = \int_0^\infty e^{-\lambda t} T_t h \, dt \quad \text{is norm convergent.}$$

1.4.2 Extension to strongly continuous group

Let $(T_t)_{t\in\mathbb{R}}$ be a strongly continuous group of bounded operators on X. Then $(T_t)_{t\geq 0}$ is a strongly continuous semigroup, and if Γ is the infinitesimal generator of $(T_t)_{t\geq 0}$, then $(S_t)_{t\geq 0} := (T_{-t})_{t\geq 0}$ is also strongly continuous semigroup with infinitesimal generator $-\Gamma$.

Thus, if $(T_t)_{t\in\mathbb{R}}$ is a strongly continuous group of bounded operators on X, then both Γ and $-\Gamma$ are infinitesimal generators of strongly continuous semigroups denoted by T_t^+ and T_{-t}^- respectively. Conversely, if Γ and $-\Gamma$ are generators of strongly continuous semigroups T_t^+ and T_t^-, then Γ is the generator of a strongly continuous group $(T_t)_{t\in\mathbb{R}}$ given by

$$T_t = \begin{cases} T_t^+ \text{ for } t \geq 0, \\ T_{-t}^- \text{ for } t \leq 0. \end{cases}$$

We refer to [30, Chapter I], or [18, Chapter II] specifically for detailed theory about extension of semigroups to groups in Banach spaces.

1.4.3 Similar semigroups

If X and Y are arbitrary Banach spaces and $U \in \mathcal{L}(X,Y)$ is an invertible operator, then clearly $(A_t)_{t\in\mathbb{R}} \subset \mathcal{L}(X)$ is a strongly continuous group if and only if $B_t := UA_tU^{-1}$, $t \in \mathbb{R}$, is a strongly continuous group in $\mathcal{L}(Y)$. In this case, if $(A_t)_{t\in\mathbb{R}}$ has generator Γ, then $(B_t)_{t\in\mathbb{R}}$ has generator $\Delta = U\Gamma U^{-1}$ with domain

$$D(\Delta) = UD(\Gamma) := \{y \in Y : Uy \in \mathcal{D}(\Gamma)\}.$$

Moreover,

$$\sigma_p(\Delta, Y) = \sigma_p(\Gamma, X),$$

and

$$\sigma(\Delta, Y) = \sigma(\Gamma, X),$$

since if λ is in the resolvent set $\rho(\Gamma, X) := \mathbb{C} \setminus \sigma(\Gamma, X)$, we have that $R(\lambda, \Delta) = UR(\lambda, \Gamma)U^{-1}$. See for example [18, Chapter II].

1.4.4 The dual semigroup

Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup on X. For $t \geq 0$, let T_t^* be the adjoint operator of T_t. From the definition of the adjoint operator, it is clear that the family $(T_t^*)_{t\geq 0}$ of bounded operators on X^* satisfies the semigroup property. This family is therefore called the adjoint or the dual semigroup of T_t.

However, the adjoint semigroup need not be strongly continuous on X^* since the mapping $T_t \to T_t^*$ does not necessarily conserve strong continuity of T_t, see [30] for example. The following result gives the relation between semigroups and adjoints with their generators.

Proposition 3 ([30, Corollary 10.6])

Let X be a reflexive Banach space and let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup on X with infinitesimal generator Γ. The adjoint semigroup T_t^* of T_t is strongly continuous semigroup on X^* whose infinitesimal generator is Γ^* the adjoint of Γ.

1.4.5 Spectral Mapping Theorem for semigroups

Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup on Banach space X, with the infinitesimal generator Γ. Of course, it is a nice idea to describe the relations between the spectrum $\sigma(T_t)$ and that of its generator $\sigma(\Gamma)$. Intuitively, one would expect the relation "$\sigma(T_t) = e^{t\sigma(\Gamma)} := \left\{e^{t\lambda} : \lambda \in \sigma(\Gamma)\right\}$." However, this is not true in general and we refer to [18, p.176] and [30, p.44] for a complete account of these facts. The following theorem outlines the true picture of this relation;

Theorem 3 (Spectral Mapping Theorem for Semigroups)

Let $(T_t)_{t\geq 0}$ be strongly continuous semigroup and let Γ be its infinitesimal generator. Then

1. $\sigma(T_t) \supset e^{t\sigma(\Gamma)}$,
2. $\sigma_p(T_t) \setminus \{0\} = e^{t\sigma_p(\Gamma)}$,
3. $\sigma_r(T_t) \setminus \{0\} = e^{t\sigma_r(\Gamma)}$.

Moreover, if $(T_t)_{t\geq 0}$ is strongly continuous semigroup of normal operators on a Hilbert space and Γ denotes its generator. Then

$$\sigma(T_t) = \overline{e^{t\sigma(\Gamma)}},$$

where $\overline{e^{t\sigma(\Gamma)}}$ is the complex conjugate of $e^{t\sigma(\Gamma)}$, see [18, Chapter V].

CHAPTER 2

HARDY AND BERGMAN SPACES OF THE UPPER HALF - PLANE

For an open subset Ω of $\mathbb{C}$, let $\mathcal{H}(\Omega)$ denote the Fréchet space of analytic functions $f : \Omega \to \mathbb{C}$ endowed with the topology of uniform convergence on compact subsets of Ω. In this chapter, we discuss basic properties of both Hardy and weighted Bergman spaces of the upper half-plane, $\mathbb{U}$. In section 2.2, we discuss the Bloch spaces while in the last section, the Szegö and Bergman projections on the upper half-plane are considered. From the existing literature, the Bergman projections on the unit disc $\mathbb{D}$ is well documented, while the corresponding theory on the half-plane is scattered. We provide a more unified and readable presentation. We also briefly consider another related type of projection on Hardy spaces called the Cauchy-Szegö projection. Consequently, we obtain the duality of both the Hardy and Bergman spaces of the half-plane.

2.1 Hardy and Bergman spaces of the disc and upper half - plane

For $1 \leq p < \infty$, the Hardy spaces of the upper half plane, $H^p(\mathbb{U})$, are defined as

$$H^P(\mathbb{U}) := \left\{ f \in \mathcal{H}(\mathbb{U}) : \|f\|_{H^p(\mathbb{U})} := \sup_{y>0} \left(\int_{-\infty}^{\infty} |f(x+iy)|^p dx \right)^{1/p} < \infty \right\},$$

while the Hardy spaces of the unit disc, $H^p(\mathbb{D})$, by

$$H^P(\mathbb{D}) := \left\{ f \in \mathcal{H}(\mathbb{D}) : \|f\|^p_{H^p(\mathbb{D})} := \sup_{0<r<1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p \, d\theta < \infty \right\}.$$

We note that every function $f \in H^p(\mathbb{U})$ (or $H^p(\mathbb{D})$) has non-tangential boundary values almost everywhere on $\partial\mathbb{U}$ (or $\partial\mathbb{D}$). In particular, H^p-functions my be identified with their boundary values and with this convention,

$$\|f\|_{H^p(\mathbb{U})} = \left(\int_{-\infty}^{\infty} |f(x)|^p \, dx\right)^{\frac{1}{p}},$$

and respectively,

$$\|f\|_{H^p(\mathbb{D})} = \left(\int_{0}^{2\pi} |f(e^{i\theta})|^p \, d\theta\right)^{\frac{1}{p}}.$$

On the other hand, for $1 \leq p < \infty$, $\alpha > -1$, the weighted Bergman spaces on the upper half plane, $L_a^p(\mathbb{U}, \mu_\alpha)$, are defined by

$$L_a^p(\mathbb{U}, \mu_\alpha) := \left\{ f \in \mathcal{H}(\mathbb{U}) : \|f\|_{L_a^p(\mathbb{U},\mu_\alpha)} = \left(\int_{\mathbb{U}} |f(z)|^p \, d\mu_\alpha(z)\right)^{\frac{1}{p}} < \infty \right\},$$

while the corresponding spaces on the disc, $L_a^p(\mathbb{D}, m_\alpha)$, by

$$L_a^p(\mathbb{D}, m_\alpha) := \left\{ f \in \mathcal{H}(\mathbb{D}) : \|f\|_{L_a^p(\mathbb{D},m_\alpha)} = \left(\int_{\mathbb{D}} |f(z)|^p \, dm_\alpha(z)\right)^{\frac{1}{p}} < \infty \right\}.$$

In particular, $L_a^p(\mathbb{U}, \mu_\alpha) = L^P(\mathbb{U}, \mu_\alpha) \cap \mathcal{H}(\mathbb{U})$ and $L_a^p(\mathbb{D}, m_\alpha) = L^P(\mathbb{D}, m_\alpha) \cap \mathcal{H}(\mathbb{D})$, where $L^p(\mathbb{U}, \mu_\alpha)$ or simply $L^p(\mu_\alpha)$ ($L^p(\mathbb{D}, m_\alpha)$ or simply $L^p(m_\alpha)$) denotes the classical Lebesgue spaces associated with the weighted measure μ_α, and respectively m_α. It is important to note that the case $\alpha = 0$ yields the (unweighted) Bergman spaces.

As noted in [3] in the case of the disc, the Hardy space $H^p(\mathbb{U})$ behaves in many ways as the limiting case of $L_a^p(\mathbb{U}, \mu_\alpha)$ as $\alpha \to -1^+$. Therefore, we shall let X denote either the Hardy space $H^p(\mathbb{U})$ or the weighted Bergman space $L_a^p(\mathbb{U}, \mu_\alpha)$, and we associate with each X, a parameter $\gamma = \frac{\alpha+2}{p}$, where $\alpha = -1$ in the case that $X = H^p(\mathbb{U})$. Also, we shall let $X(\mathbb{D})$ denote the corresponding space of analytic functions on the unit disc $\mathbb{D}$. Therefore, we formulate the growth conditions for Hardy and Bergman spaces simultaneously in the next two results; while known, we provide elementary proofs.

Lemma 1

Let $X(\mathbb{D})$ denote either $H^p(\mathbb{D})$ or $L^p_a(m_\alpha)$, $1 \leq p < \infty$ and $\alpha > -1$. Let $\gamma = \frac{\alpha+2}{p}$ ($\alpha = -1$ in case $X(\mathbb{D}) = H^p(\mathbb{D})$). Then there exists a constant $C = C_{X(\mathbb{D})}$ such that for every $f \in X(\mathbb{D})$ and $z \in \mathbb{D}$,

$$|f(z)| \leq \frac{C\|f\|_{X(\mathbb{D})}}{(1-|z|^2)^\gamma}. \tag{2.1}$$

Proof: We begin by showing that $|f(0)| \leq C\|f\|$. Let $f \in H^p(\mathbb{D})$. Then $\forall\, r$, $0 < r < 1$, the mean value property implies that $f(0) = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})\, d\theta$. Thus $|f(0)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|\, d\theta$ and Jensen's inequality implies

$$|f(0)|^p \leq \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p\, d\theta \leq \|f\|^p_{H^p(\mathbb{D})}.$$

Similarly, if $f \in L^p_a(m_\alpha)$, then $\forall\, r$, $0 < r < 1$, $|f(0)|^p \leq \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p\, d\theta$. Thus

$$|f(0)|^p \int_0^1 (1-r^2)^\alpha 2r\, dr \leq \int_0^1 (1-r^2)^\alpha 2r\, dr \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p\, d\theta = \|f\|_{L^p_a(m_\alpha)}.$$

If $a \in \mathbb{D}$, let $\phi_a(z) = \frac{a-z}{1-\bar{a}z} \in \mathrm{Aut}(\mathbb{D})$, where $\mathrm{Aut}(\mathbb{D})$ denotes the group of automorphisms of $\mathbb{D}$. Then $S_{\phi_a} f := (\phi_a')^\gamma f \circ \phi_a$ is an isometry on $X(\mathbb{D})$.

Indeed, in the Hardy space case,

$$\|S_{\phi_a} f\|^p_{H^p(\mathbb{D})} = \int_0^{2\pi} |f(\phi_a(e^{i\theta}))|^p |\phi_a'(e^{i\theta}) i e^{i\theta}|\, d\theta = \int_0^{2\pi} |f(e^{it})|^p\, dt.$$

In the Bergman space case, we note that, by the Schwarz-Pick Lemma [20, Lemma I.1.2], $(1-|z|^2)|\phi_a'(z)| = 1-|\phi_a(z)|^2 \ \forall z \in \mathbb{D}$, and therefore a change of variables argument implies

$$\begin{aligned}\|S_{\phi_a} f\|^p_{L^p_a(m_\alpha)} &= \int_{\mathbb{D}} |f(\phi_a(z))|^p \left((1-|z|^2)|\phi_a'|\right)^\alpha |\phi_a'| \, dA(z) \\ &= \int_{\mathbb{D}} |f(\omega)|^p (1-|\omega|^2)^\alpha \, dA(\omega) = \|f\|^p_{L^p_a(m_\alpha)}.\end{aligned}$$

Thus if $a \in \mathbb{D}$, $|\phi_a'(0)|^\gamma |f(a)| = |S_{\phi_a} f(0)| \leq C\|f\|$ or $|f(a)| \leq \frac{C\|f\|}{(1-|a|)^\gamma}$, as claimed.

The following is an immediate consequence of the above Lemma,

Corollary 1

Let X denote either $H^p(\mathbb{U})$ or $L^p_a(\mu_\alpha)$, $1 \leq p < \infty$ and $\alpha > -1$. Let $\gamma = \frac{\alpha+2}{p}$ ($\alpha = -1$ in case $X = H^p(\mathbb{U})$). Then there exists a constant $C = C_X$ such that for every $f \in X$ and $z \in \mathbb{U}$,

$$|f(z)| \leq \frac{C\|f\|_X}{(\Im(z))^\gamma}. \tag{2.2}$$

Proof: Let $g = S_\psi f$. Then by Proposition 10 in Chapter 3, $\|g\|_{H^p(\mathbb{D})} = \|f\|_{H^p(\mathbb{U})}$ and $\|g\|_{L^p_a(m_\alpha)} = 2^{-\frac{\alpha}{p}} \|f\|_{L^p_a(\mu_\alpha)}$. Now, if $a = \psi^{-1}(z)$, then $(1-|a|^2)|\psi'(a)| = 2\Im(z)$ and

$$|\psi'(a)|^\gamma |f(z)| \leq |g(a)| \leq \frac{C\|f\|}{(1-|a|^2)^\gamma},$$

implying that $|f(z)| \leq \frac{C\|f\|}{(2\Im(z))^\gamma}$.

As consequence of the growth conditions given by equations (2.1) and (2.2), both Hardy and Bergman spaces (X and $X(\mathbb{D})$) are Banach spaces. In fact, if $p = 2$, it turns out that these spaces are Hilbert spaces, and moreover, the Bergman space L^p_a is a closed subspace of the classical Lebesgue space L^p. For a detailed theory of Hardy spaces, we refer to [15, 20, 31], while for Bergman spaces, see [8, 16, 21, 31, 41].

2.2 Bloch spaces of the disc and upper half-plane

The Bloch space of the unit disc, denoted by $\mathcal{B}_\infty(\mathbb{D})$, is defined to be the space of analytic functions f on $\mathbb{D}$ such that the seminorm

$$\|f\|_{\mathcal{B}_{\infty,1}(\mathbb{D})} = \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty.$$

Following [5], $\mathcal{B}_\infty(\mathbb{D})$ is a Banach space with respect to the norm

$$\|f\|_{\mathcal{B}_\infty(\mathbb{D})} := |f(0)| + \|f\|_{\mathcal{B}_{\infty,1}(\mathbb{D})} < \infty.$$

The corresponding Bloch space of the upper half plane denoted by $\mathcal{B}_\infty(\mathbb{U})$ is easily obtained through the Cayley transformation $\psi : \mathbb{D} \to \mathbb{U}$, and is defined by

$$\mathcal{B}_\infty(\mathbb{U}) := \left\{ f \in \mathcal{H}(\mathbb{U}) : \sup_{z\in\mathbb{U}} \Im(z)|f'(z)| < \infty \right\},$$

with the seminorm

$$\|f\|_{\mathcal{B}_{\infty,1}(\mathbb{U})} := \sup_{z\in\mathbb{U}} \Im(z)|f'(z)|,$$

and norm

$$\|f\|_{\mathcal{B}_\infty(\mathbb{U})} := |f(i)| + \|f\|_{\mathcal{B}_{\infty,1}(\mathbb{U})}.$$

On the other hand, the Little Bloch space of the disc is denoted by $\mathcal{B}_{\infty,0}(\mathbb{D})$, and defined to be the closed subspace of $\mathcal{B}_\infty(\mathbb{D})$ such that

$$\mathcal{B}_{\infty,0}(\mathbb{D}) := \mathrm{cl}_{\mathcal{B}_\infty}\, \mathbb{C}[z],$$

where $\mathbb{C}[z]$ denotes the analytic polynomials in z. Equivalently,

$$\mathcal{B}_{\infty,0}(\mathbb{D}) := \left\{ f \in \mathcal{H}(\mathbb{D}) : \lim_{|z|\to 1^-, z\in\mathbb{D}} (1-|z|^2)|f'(z)| = 0 \right\},$$

and possesses the same norm as $\mathcal{B}_\infty$, see for example [40, Section 5.2].

The corresponding Little Bloch space of the upper half plane denoted by $\mathcal{B}_{\infty,0}(\mathbb{U})$, is also easily obtained via the Cayley transform ψ and is defined by

$$\mathcal{B}_{\infty,0}(\mathbb{U}) := \left\{ f \in \mathcal{H}(\mathbb{U}) : \lim_{\Im(z)\to 0, z\in\mathbb{U}} \Im(z)|f'(z)| = 0 \right\},$$

with the same norm as $\mathcal{B}_\infty(\mathbb{U})$. For a detailed theory on Bloch spaces, we refer to [20], [40] or [41].

2.3 Bergman projections of the upper half - plane

We begin this section by giving the following definition: Let H denote a Hilbert space of functions defined on an open set $\Omega \subset \mathbb{C}$. We call a reproducing kernel for H, a complex function $K : \Omega \times \Omega \to \mathbb{C}$ such that, if we put $K_\omega(z) := K(z, \omega)$, then the following two properties hold:

1. for every $\omega \in \Omega$, the function K_ω belongs to H, and
2. for all $f \in H$ and $\omega \in \Omega$, we have

$$f(\omega) = \langle f, K_\omega \rangle.$$

It is clear that the above two properties imply that such a kernel K satisfies the identity $K(z,\omega) = \overline{K(\omega, z)}$ for all $z, \omega \in \Omega$. Indeed

$$\begin{aligned} K(z,\omega) &= K_\omega(z) = \langle K_\omega, K_z \rangle \\ &= \overline{\langle K_z, K_\omega \rangle} = \overline{K_z(\omega)} = \overline{K(\omega, z)}. \end{aligned}$$

The growth condition estimates (for X and $X(\mathbb{D})$) given by equations (2.1) and (2.2) and the Riesz representation theorem for H^*, imply that H^2 and L^2_a are reproducing kernel

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surjective isomorphism of $L_a^2(\mathbb{U},\mu_\alpha)$ onto $L_a^2(\mathbb{D},m_\alpha)$. Since L_a^2 is a Hilbert space, it then follows that T is unitary, that is, $T^* = T^{-1}$.

For every $\xi \in \mathbb{D}$, and by writing $K_{\alpha,\mathbb{D},\xi} = K_{\alpha,\mathbb{D}}(\cdot,\xi)$ simply as $K_{\mathbb{D},\xi}$, we have using the definition of K_α,

$$Tf(\xi) = 2^{-\frac{\alpha}{2}}(\psi'(\xi))^{1+\frac{\alpha}{2}} f(\psi(\xi)) = 2^{-\frac{\alpha}{2}}\frac{(2i)^{1+\frac{\alpha}{2}}}{(1-\xi)^{2+\alpha}} f(\psi(\xi)),$$
$$= \langle Tf, K_{\mathbb{D},\xi}\rangle_{L_a^2(\mathbb{D},m_\alpha)} = \langle f, T^{-1}K_{\mathbb{D},\xi}\rangle_{L_a^2(\mathbb{U},\mu_\alpha)}.$$

We now compute $T^{-1}K_{\mathbb{D},\xi}$. For $z \in \mathbb{U}$, we have

$$\left(T^{-1}K_{\mathbb{D},\xi}\right)(z) = 2^{\frac{\alpha}{2}}\left((\psi^{-1}(z))'\right)^{1+\frac{\alpha}{2}} K_{\mathbb{D},\xi}\left(\psi^{-1}(z)\right).$$

But by equation (2.3),

$$K_{\mathbb{D},\xi}\left(\psi^{-1}(z)\right) = \frac{1}{\left(1-\frac{z-i}{z+i}\bar{\xi}\right)^{2+\alpha}} = \frac{(z+i)^{2+\alpha}}{(1-\bar{\xi})^{2+\alpha}[z-\overline{\psi(\xi)}]}.$$

Therefore,

$$2^{-\frac{\alpha}{2}}(\psi'(\xi))^{1+\frac{\alpha}{2}} f(\psi(\xi)) = Tf(\xi) = \left\langle f, T^{-1}K_{\mathbb{D},\xi}\right\rangle_{L_a^2(\mathbb{U},\mu_\alpha)}$$
$$= 2^{\frac{\alpha}{2}}\frac{(-2i)^{1+\frac{\alpha}{2}}}{(1-\bar{\xi})^{2+\alpha}}\left\langle f, \frac{1}{(z-\overline{\psi(\xi)})^{2+\alpha}}\right\rangle,$$

which implies that $f(\psi(\xi)) = 2^\alpha\left\langle f, \frac{1}{[-i(z-\overline{\psi(\xi)})]^{2+\alpha}}\right\rangle$, and thus,

$$f(\omega) = \left\langle f, \frac{2^\alpha}{[-i(z-\overline{\omega})]^{2+\alpha}}\right\rangle, \quad \omega \in \mathbb{U}.$$

In particular, if we write $\psi(\xi) = \omega$, then

$$K_{\mathbb{U},\omega}(z) = \frac{2^\alpha}{[-i(z-\overline{\omega})]^{2+\alpha}}, \quad \text{as desired.}$$

∎

It is important to note that a similar formula has been obtained through the use of other methods, but most prominently, the use of Paley-Weiener theorem which in itself involves Fourier transform, see [8]. Since $L_a^2(\mathbb{U}, \mu_\alpha)$ is a closed subspace of the Hilbert space $L^2(\mathbb{U}, \mu_\alpha)$, there exists an orthogonal projection $P_\alpha : L^2(\mathbb{U}, \mu_\alpha) \to L_a^2(\mathbb{U}, \mu_\alpha)$ which we shall call the weighted Bergman projection on $L^2(\mathbb{U}, \mu_\alpha)$.

Proposition 4

The weighted Bergman kernel $K_\alpha(\cdot, \omega)$ from $L^2(\mathbb{U}, \mu_\alpha)$ onto the subspace $L_a^2(\mathbb{U}, \mu_\alpha)$ is given explicitly by

$$P_\alpha f(z) = \int_{\mathbb{U}} K_\alpha(z, \omega) f(\omega)\, d\mu_\alpha(\omega), \tag{2.5}$$

where K_α is the weighted Bergman kernel the half-plane given by equation (2.4).

Proof: Indeed, by the reproducing property of $K_\alpha(z, \omega)$ and the self - adjointness of P_α on $L^2(\mathbb{U}, \mu_\alpha)$, we have

$$\begin{aligned} P_\alpha f(z) = \langle P_\alpha f, K_\alpha(., z)\rangle_{L_a^2(\mu_\alpha)} \ &= \ \langle f, P_\alpha K_\alpha(., z)\rangle_{L_a^2(\mu_\alpha)} \\ = \langle f, K_\alpha(., z)\rangle_{L_a^2(\mu_\alpha)} \ &= \ \int_{\mathbb{U}} K_\alpha(z, \omega) f(\omega)\, d\mu_\alpha(\omega), \quad \text{as claimed.} \end{aligned}$$

∎

At this point, it is natural to ask whether the Bergman projection P_α extends in some meaningful way to $L_a^p(\mathbb{U}, \mu_\alpha)$ for the case $p \neq 2$, and in that case, whether the reproducing property of $K_\alpha(z, \omega)$, (that is, $P_\alpha F = F$) holds in $L_a^p(\mathbb{U}, \mu_\alpha)$. These problems were posed in [8]. In this section, we address the above questions but first we prove some elementary results that will be useful in the sequel.

Proposition 5

Let $X = H^p(\mathbb{U})$ *or* $L_a^p(\mathbb{U}, \mu_\alpha)$, $1 \leq p < \infty$. *Let* $a > 0$, $b \in \mathbb{R}$ *and define* $Tf(z) = f(az+b)$ *for every* $f \in X$, *then* $\|T\| \leq a^{-\gamma}$.

Proof: If $f \in L_a^p(\mathbb{U}, \mu_\alpha)$, then

$$\begin{aligned} \|Tf\|_p^p &= \int_{\mathbb{U}} |f(az+b)|^p \frac{(\Im(az+b))^\alpha}{a^{\alpha+2}} |a|^2 \, dA(z) \\ &= \int_{a\mathbb{U}} |f(\omega)|^p (\Im(\omega))^\alpha \, dA(\omega) \, a^{-(\alpha+2)}, \end{aligned}$$

and if $f \in H^p(\mathbb{U})$, then

$$\begin{aligned} \|Tf\|_p^p &= \sup_{y>0} \int_{-\infty}^{\infty} |f(ax+iay+b)|^p \, dx = \sup_{t>0} \int_{-\infty}^{\infty} |f(ax+b+it)|^p \, dx \\ &= \frac{1}{a} \sup_{t>0} \int_{-\infty}^{\infty} |f(s+it)|^p \, ds = \frac{1}{a} \|f\|_p^p. \end{aligned}$$

■

The next two Lemmas give examples of analytic functions and the conditions they must satisfy to belong to the spaces X and $X(\mathbb{D})$.

Lemma 2

Let $X(\mathbb{D})$ *denote one of the spaces* $H^p(\mathbb{D})$ *or* $L_a^p(\mathbb{D}, m_\alpha)$, $1 \leq p < \infty$ *and* $\alpha > -1$ ($\alpha = -1$ *if* $X(\mathbb{D}) = H^p(\mathbb{D})$), *and let* $\gamma = (\alpha+2)/p$. *Then for* $\eta \in \mathbb{C}$,

$$(e^{i\theta} - z)^\eta \in X(\mathbb{D}) \text{ if and only if } \Re\eta > -\gamma.$$

Proof: We first consider the Bergman space case, that is $X(\mathbb{D}) = L^p_a(\mathbb{D}, m_\alpha)$.

Recall $(e^{i\theta} - z)^\eta \in L^p_a(\mathbb{D}, m_\alpha) \Leftrightarrow \int_{\mathbb{D}} |(e^{i\theta} - z)^\eta|^p \, dm_\alpha(z) < \infty$. Now,

$$\begin{aligned}\int_{\mathbb{D}} |(e^{i\theta} - z)^\eta|^p \, dm_\alpha(z) &= \int_{\mathbb{D}} |(e^{i\theta} - z)^\eta|^p (1 - |z|^2)^\alpha \, dA(z) \\ &= \int_{\mathbb{D}} |1 - ze^{-i\theta}|^{p\Re(\eta)} (1 - |z|^2)^\alpha \, dA(z) \\ &= \int_{\mathbb{D}} \frac{(1 - |z|^2)^\alpha}{|1 - ze^{-i\theta}|^{-p\Re(\eta)}} \, dA(z). \qquad (2.6)\end{aligned}$$

It then follows immediately from [41, Lemma 3.10] that equation (2.6) is bounded if and only if $-p\Re(\eta) - \alpha - 2 < 0$, that is, $\Re(\eta) > -\frac{\alpha+2}{p}$, as desired.

For $X(\mathbb{D}) = H^p(\mathbb{D})$, we use the fact that functions in $H^p(\mathbb{D})$ can be identified with their boundary values. Fix $\theta \in \mathbb{R}$ and let $f(z) = (e^{i\theta} - z)^\eta$. Then f has boundary values $f(e^{it}) = (e^{i\theta} - e^{it})^\eta$ and $f \in H^p(\mathbb{D})$ is equivalent to

$$\int_{-\pi}^{\pi} |f(e^{it})|^p \, dt = \int_{\theta-\pi}^{\theta+\pi} \left|(1 - e^{i(t-\theta)})^\eta\right|^p \, dt = \int_{-\pi}^{\pi} 2^{p\Re(\eta)} \left|\sin(t/2)\right|^{p\Re(\eta)} \, dt < \infty. \quad (2.7)$$

But the equation (2.7) holds if and only if $p\Re(\eta) > -1$, as claimed. ∎

Lemma 3

Let X denote one of the spaces $H^p(\mathbb{U})$ or $L^p_a(\mathbb{U}, \mu_\alpha)$, $1 \leq p < \infty$ and $\alpha > -1$ ($\alpha = -1$ if $X = H^p(\mathbb{U})$), and let $\gamma = (\alpha + 2)/p$. If $c \in \mathbb{R}$ and $\lambda,\ \nu \in \mathbb{C}$, then

1. *$f(\omega) = (\omega - c)^\lambda (\omega + i)^\nu \in X$ if and only if $\Re(\lambda + \nu) < -\gamma < \Re(\lambda)$. In particular, $(\omega - c)^\lambda \notin X$ for any $\lambda \in \mathbb{C}$, and $(\omega + i)^\nu \in X$ if and only if $\Re\nu < -\gamma$.*
2. *$f(\omega) = e^{i\omega}/\omega^c \in X$ if and only if $1/p < c < \gamma$. In particular, $e^{i\omega}/\omega^c \notin H^p(\mathbb{U})$ for any $c \in \mathbb{R}$.*

Proof: Let $f(\omega) = (\omega - c)^\lambda (\omega + i)^\nu$. Then by Proposition 10, $f \in X$ if and only if $S_\psi f \in X(\mathbb{D})$, the corresponding space of analytic functions on $\mathbb{D}$. By direct computation, we have

$(\psi'(z))^\gamma = \left(\frac{2i}{(1-z)^2}\right)$, and $f(\psi(z)) = (\psi(z)-c)^\lambda(\psi(z)+i)^\nu = \left(\frac{c+i\left(z-\frac{c-i}{c+i}\right)}{1-z}\right)^\lambda \left(\frac{2i}{1-z}\right)^\nu$.

Thus

$$\begin{aligned} S_\psi f(z) &= \frac{(2i)^\gamma}{(1-z)^{2\gamma}} \cdot \frac{(c+i)^\lambda}{(1-z)^\lambda} \cdot \left(z - \frac{c-i}{c+i}\right)^\lambda \cdot \frac{(2i)^\nu}{(1-z)^\nu} \\ &= K\frac{\left(z-\frac{c-i}{c+i}\right)^\lambda}{(1-z)^{2\gamma+\lambda+\nu}}, \quad \text{where } K \neq 0 \text{ is a constant.} \end{aligned}$$

Since $\frac{c-i}{c+i} \in \partial\mathbb{D}$, there exists $\theta \in \mathbb{R}$ such that $\frac{c-i}{c+i} = e^{i\theta}$. From Lemma 2 above, $S_\psi f \in X(\mathbb{D})$ if and only if $\Re(\lambda) > -\gamma$ and $\Re(-2\gamma-\lambda-\nu) > -\gamma$; that is, $\Re(\lambda+\nu) < -\gamma < \Re(\lambda)$, as claimed.

The other two assertions of (1) are special cases of the first claim proved above and follow easily by taking $\nu = 0$ and $\lambda = 0$ respectively.

Assertion (2) of the Lemma follows from (1) in the case that $X = H^p(\mathbb{U})$ by identifying f and its boundary values on $\mathbb{R}$. The function $f(\omega) = e^{i\omega}/\omega^c$ is analytic on $\mathbb{U}$ for all $c \in \mathbb{R}$, and letting $\omega = x + iy$,

$$\int_{\mathbb{U}} |f(\omega)|^p \, d\mu_\alpha(\omega) = \int_0^\infty e^{-yp} y^{\alpha-cp+1} \, dy \int_{-\infty}^\infty \frac{1}{(u^2+1)^{cp/2}} \, du.$$

The first integral converges if and only if $c < \gamma$, and the second converges if and only if $cp > 1$. Combining the two we obtain the result. ■

We now give the following proposition,

Proposition 6

Let $X = H^p(\mathbb{U})$ or $L^p_a(\mathbb{U}, \mu_\alpha)$. If $v > 0$, then

$$(\omega + iv)^\nu \in X \text{ if and only if } \nu > -\gamma.$$

Proof: If $f(\omega) = (\omega + iv)^\nu$, then

$$f(\omega) = v^\nu \left(\frac{1}{v}\omega + i\right)^\nu = v^\nu Tg(\omega),$$

where $Th(z) = h\left(\frac{1}{v}z\right)$ and $g(\omega) = (\omega+i)^\nu$. Now Proposition 5 and Lemma 3 immediately yield the desired result. ■

To begin addressing the questions mentioned earlier in this section concerning the extension of the Bergman kernel K_α to the cases $p \neq 2$, we give the following direct consequence of the above proposition.

Corollary 2

For fixed $\omega \in \mathbb{U}$, the Bergman kernel $K_\alpha(\cdot, \omega)$ belongs to $L^q_a(\mathbb{U}, \mu_\alpha)$ if and only if $1 < q \leq \infty$.

Proof: If $K_\alpha(z, \omega) = \frac{2^\alpha}{(-i(z-\overline{\omega}))^{\alpha+2}}$, $(z, \omega \in \mathbb{U})$, then for fixed $\omega \in \mathbb{U}$,

$$K_\alpha(z, \omega) = 2^\alpha i^{\alpha+2} \left((z - \Re(\omega)) + i\Im(\omega)\right)^{-(\alpha+2)}.$$

Therefore by Proposition 6, $K_\alpha(\cdot, \omega) \in L^q_a(\mathbb{U}, \mu_\alpha)$ if and only if $-(\alpha+2) < -(\alpha+2)/q$, which is equivalent to $q > 1$. Moreover, if, $z = x + iy$, $y > 0$, we have

$$|K_\alpha(z, \omega)| \leq \frac{2^\alpha}{(\Im(\omega))^{\alpha+2}},$$

implying that $K_\alpha(\cdot, \omega) \in L^\infty_a(\mathbb{U}, \mu_\alpha)$. ■

We can now prove the following result;

Proposition 7

Let $1 \leq p < \infty$, then for each $f \in L^p_a(\mathbb{U}, \mu_\alpha)$,

$$f(z) = \int_{\mathbb{U}} f(\omega) K_\alpha(z, \omega)\, d\mu_\alpha(\omega).$$

Proof: If $f \in L_a^p(\mathbb{U}, \mu_\alpha)$, $1 \leq p < \infty$, then by Corollary 2, $K_\alpha(\cdot, z) \in L_a^p(\mu_\alpha)$, $\frac{1}{p} + \frac{1}{q} = 1$ and so Hölder's inequality implies that

$$f \longmapsto \int_{\mathbb{U}} f(\omega) K_\alpha(z, \omega)\, d\mu_\alpha(\omega) = \langle f, K_\alpha(\cdot, z)\rangle$$

is continuous; moreover, by the reproducing property,

$$f(z) = \int_{\mathbb{U}} f(\omega) K_\alpha(z, \omega)\, d\mu_\alpha(\omega) \text{ for all } f \in L_a^p(\mu_\alpha) \cap L_a^2(\mu_\alpha).$$

Since $L_a^p(\mu_\alpha) \cap L_a^2(\mu_\alpha)$ is dense in $L_a^p(\mu_\alpha)$, we're done. ∎

The following theorem characterizes when P_α is a bounded projection from $L^p(\mathbb{U}, \mu_\alpha)$ onto $L_a^p(\mathbb{U}, \mu_\alpha)$, see D. Békollé, et.al. [8] for the details.

Theorem 5

The Bergman projection

$$P_\alpha f(z) := \int_{\mathbb{U}} f(\omega) K_\alpha(z, \omega)\, d\mu_\alpha(\omega), \quad \alpha > -1,$$

is a bounded projection from $L^p(\mathbb{U}, \mu_\alpha)$ onto $L_a^p(\mathbb{U}, \mu_\alpha)$ if and only if $1 < p < \infty$.

An immediate consequence of the boundedness of the Bergman projection P_α on $L_a^p(\mathbb{U}, \mu_\alpha)$ is the duality of Bergman spaces $L_a^p(\mathbb{U}, \mu_\alpha)$ which we give in the following result,

Corollary 3

Let $1 < p < \infty$ and q be conjugate to p in the sense that $\frac{1}{p} + \frac{1}{q} = 1$. Let $(L_a^p(\mathbb{U}, \mu_\alpha))^*$ be the dual space of $L_a^p(\mathbb{U}, \mu_\alpha)$, then

$$(L_a^p(\mathbb{U}, \mu_\alpha))^* \approx L_a^q(\mathbb{U}, \mu_\alpha), \quad \alpha > -1, \tag{2.8}$$

under the sesquilinear pairing

$$\langle f, g\rangle = \int_{\mathbb{U}} f(\omega)\overline{g(\omega)}\, d\mu_\alpha \quad (f \in L_a^p(\mu_\alpha),\ g \in L_a^q(\mu_\alpha)). \tag{2.9}$$

Proof: The classical duality between L^p- spaces gives

$$(L^p(\mathbb{U}, \mu_\alpha))^* \approx L^q(\mathbb{U}, \mu_\alpha).$$

By Hahn-Banach extension theorem and the boundedness of the Bergman projection P_α for $1 < p < \infty$, (Theorem 5), we have

$$(L_a^p(\mathbb{U}, \mu_\alpha))^* = P_\alpha(L^p(\mathbb{U}, \mu_\alpha))^* \approx P_\alpha L^q(\mathbb{U}, \mu_\alpha) = L_a^q(\mathbb{U}, \mu_\alpha), \quad \text{as desired.}$$

■

It is important to note from Theorem 5 that the Bergman projection $P_\alpha : L^1(\mathbb{U}, \mu_\alpha) \to L_a^1(\mathbb{U}, \mu_\alpha)$ is unbounded and therefore its analysis has to be considered independently. Take note that under the above duality pairing, see equation (2.9), the adjoint operator is conjugate linear. Moreover, $L_a^p(\mu_\alpha)$ spaces for $1 < p < \infty$ are reflexive and thus;

$$(L_a^q(\mu_\alpha))^* \approx (L_a^p(\mu_\alpha))^{**} \approx L_a^p(\mu_\alpha).$$

2.3.1 The Cauchy-Szegö projection

We refer to [16] or [23, Chapter 8] for a good account of the theory of the Szegö kernel and projection on Hardy spaces. Recall that functions in $H^P(\mathbb{D})$ have boundary values almost everywhere in $L_a^p(\partial\mathbb{D})$ and that if $1 \leq p < \infty$, then $H^p(\mathbb{D}) = \mathrm{cl}_{L^p(\partial\mathbb{D})}\mathbb{C}[z]$, where $\mathbb{C}[z]$ denotes analytic polynomials in z. In fact $H^2(\mathbb{D})$ has orthonormal basis $(z^n)_{n\geq 0}$.

As noted in [16, Chapter 2], Cauchy's theorem implies that the reproducing kernel for $H^1(\mathbb{D})$ is given by

$$\mathcal{S}_{\mathbb{D}}(z, \omega) = \frac{1}{1 - z\bar{\omega}} \quad (z \in \partial\mathbb{D},\ \omega \in \mathbb{D}). \tag{2.10}$$

The reproducing kernel $\mathcal{S}_{\mathbb{D}}$ is also known as the Szegö or Cauchy-Szegö kernel on the disc. Thus, the Szegö projection $P_{\mathbb{D}}$ is given by

$$\begin{aligned} P_{\mathbb{D}}\varphi(z) = \langle \varphi, \mathcal{S}(\cdot, z)\rangle &= \int_{\partial\mathbb{D}} \varphi(\omega)\overline{\mathcal{S}_{\mathbb{D}}(\omega, z)}\, dm(\omega) \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\varphi(e^{it})}{1-e^{-it}z}\, dt, \end{aligned}$$

and satisfies $P_{\mathbb{D}}f = f$ for all $f \in H^1(\mathbb{D})$. The following theorem characterizes the boundedness of the Szegö projection for the case when $p \neq 1$.

Theorem 6

If $1 < p < \infty$, then $P_{\mathbb{D}} : L^p(\partial\mathbb{D}) \to H^p(\mathbb{D})$ is bounded and surjective.

For details, see [16, Chapter 2].

Let $T : H^p(\mathbb{U}) \to H^p(\mathbb{D})$ be given by $Tf(z) = (\psi'(z))^{1/p} f(\psi(z))$. Then by Proposition 10, T is surjective isometry. In particular, $T : H^2(\mathbb{U}) \to H^2(\mathbb{D})$ is unitary with $T^* = T^{-1}$, and $T^{-1}g(\omega) = ((\psi^{-1})'(\omega))^{1/p}\, g(\psi^{-1}(\omega))$, where ψ is the Cayley transform.

We wish to compute the corresponding Szegö kernel and projection on the upper half-plane, $\mathbb{U}$: Let $\xi \in \mathbb{U}$, $z = \psi^{-1}(\xi) \in \mathbb{D}$. Also, let $f \in H^2(\mathbb{U})$ and $g = Tf$. Then

$$\begin{aligned} g(z) &= \int_{\partial\mathbb{D}} g(\omega)\mathcal{S}_{\mathbb{D}}(z, \omega)\, dm(\omega) = \langle g, \mathcal{S}(\cdot, z)\rangle_{\partial\mathbb{D}} = \langle Tf, \mathcal{S}_{\mathbb{D},z}\rangle_{\partial\mathbb{D}} \\ &= \langle f, T^*\mathcal{S}_{\mathbb{D},z}\rangle_{\mathbb{R}}. \end{aligned}$$

But we have

$$g(z) = g(\psi^{-1}(\xi)) = Tf(\psi^{-1}(\xi)) = \left(\psi'(\psi^{-1}(\xi))\right)^{1/p} f(\xi) = \frac{\xi + i}{(2i)^{1/2}},$$

and

$$\begin{aligned}\left(T^{*}\mathcal{S}_{\mathbb{D},z}\right)(\omega) &= \left((\psi^{-1})'(\omega)\right)^{1/2}\mathcal{S}_{\mathbb{D}}\left(\psi^{-1}(\omega),\psi^{-1}(\xi)\right)\\ &= \frac{(2i)^{1/2}}{(\omega+i)}\left(\frac{1}{1-\left(\frac{\omega-1}{\omega+i}\right)\left(\frac{\bar{\xi}+i}{\bar{\xi}-i}\right)}\right) = \frac{(2i)^{1/2}(\bar{\xi}-i)}{-2i(\omega-\bar{\xi})}.\end{aligned}$$

Thus

$$f(\xi) = \frac{(2i)^{1/2}}{(\xi+i)}\int_{\mathbb{R}} f(x)\overline{\left(\frac{(2i)^{1/2}(\bar{\xi}-i)}{2(-i)(x-\bar{\xi})}\right)}\,dx = \int_{\mathbb{R}} f(x)\overline{\left(\frac{(2i)^{1/2}(-2i)^{1/2}}{2(-i)(x-\bar{\xi})}\right)}\,dx.$$

Therefore, for every $f \in H^2(\mathbb{U})$, $\xi \in \mathbb{U}$,

$$f(\xi) = \int_{\mathbb{R}} f(x)\overline{\left(\frac{i}{x-\bar{\xi}}\right)}\,dx.$$

Thus the Szegö kernel for $H^2(\mathbb{U})$ is given by

$$\mathcal{S}_{\mathbb{U}}(z,\xi) = \frac{i}{z-\bar{\xi}}, \tag{2.11}$$

and hence the Szegö projection P on the $\mathbb{U}$ becomes

$$\begin{aligned}P\varphi(\xi) &= \int_{\mathbb{R}} \varphi(x)\overline{\mathcal{S}_{\mathbb{U}}(x,\xi)}\,dx = \int_{\mathbb{R}} \varphi(x)\left(\frac{-i}{x-\xi}\right)\,dx\\ &= \int_{\mathbb{R}} \varphi(x)\frac{i}{\xi-x}\,dx = \int_{\mathbb{R}} f(x)\mathcal{S}_{\mathbb{U}}(\xi,x)\,dx.\end{aligned}$$

Therefore, the upper half-plane analogue of Theorem 6 is the following,

Theorem 7

If $f \in H^p(\mathbb{U})$, $1 \leq p < \infty$, then for every $\xi \in \mathbb{U}$,

$$f(\xi) = \int_{\mathbb{R}} f(x)\mathcal{S}_{\mathbb{U}}(\xi,x)\,dx,$$

and if $p > 1$, then the Szegö projection $P : L^p(\mathbb{R}) \to H^p(\mathbb{U})$ given by

$$P\varphi(\xi) = \int_{\mathbb{R}} \varphi(x)\mathcal{S}_{\mathbb{U}}(\xi,x)\,dx$$

is bounded and surjective.

The boundedness of the Szegö projection P on $L^p(\mathbb{R})$ given by Theorem 7 immediately yields the following duality of Hardy spaces $H^p(\mathbb{U})$, for $1 < p < \infty$.

Corollary 4

Let $1 < p < \infty$ and q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $(H^p(\mathbb{U}))^*$ be the dual space of $H^p(\mathbb{U})$. Then

$$(H^p(\mathbb{U}))^* \approx H^q(\mathbb{U}), \tag{2.12}$$

via the sequilinear pairing

$$\langle f, g \rangle = \int_{\mathbb{R}} f(x)\overline{g(x)}\, dx \quad (f \in H^p(\mathbb{U}),\, g \in H^q(\mathbb{U})). \tag{2.13}$$

Proof: It is well known that

$$(L^p(\mathbb{R}))^* \approx L^q(\mathbb{R}).$$

Now, the Hahn-Banach extension theorem together with the boundedness of the Szegö projection P for $1 < p < \infty$ will yield,

$$(H^p(\mathbb{U}))^* \approx (H^p(\mathbb{R}))^* = P(L^p(\mathbb{R}))^* \approx PL^q(\mathbb{R}) = H^q(\mathbb{R}) \approx H^q(\mathbb{U}).$$

■

Again, we take note that under the pairing in equation (2.13), the adjoint operator from $\mathcal{L}(X)$ to $\mathcal{L}(X^*)$ is also conjugate linear. Since the Hardy spaces $H^p(\mathbb{U})$, $1 < p < \infty$, are reflexive Banach spaces, it follows that

$$(H^q(\mathbb{U}))^* \approx (H^p(\mathbb{U}))^{**} \approx H^p(\mathbb{U}).$$

CHAPTER 3

AUTOMORPHISM GROUPS AND ASSOCIATED GROUP ISOMETRIES

In this chapter, groups of automorphisms of the upper half-plane are discussed. We characterize all continuous, one-parameter groups of automorphisms of the half-plane into three types and introduce the associated groups of weighted composition operators on both the Hardy and Bergman spaces. This is done in section 3.2. We also give examples that illustrate our characterization, and end this chapter by proving strong continuity property of these groups of isometries.

3.1 Automorphisms of the upper half - plane, $\mathrm{Aut}(\mathbb{U})$

In this section, we present basic properties of automorphism groups of the upper half-plane, $\mathrm{Aut}(\mathbb{U})$. For a detailed theory on $\mathrm{Aut}(\mathbb{U})$, we refer to [2], [10], or any standard complex analysis text, but we give certain basics that are of interest in this study.

The first application of the Schwarz's lemma is the computation of the automorphism group of the unit disc $\mathbb{D}$. In fact, it follows from the Schwarz's lemma that all automorphisms of $\mathbb{D}$ are maps of the form $f : \mathbb{D} \to \mathbb{D}$ such that

$$f(z) = e^{i\theta} \frac{z-a}{1-\overline{a}z}, \tag{3.1}$$

for some $\theta \in \mathbb{R}$, $a = f^{-1}(0) \in \mathbb{D}$, or equivalently of the form $f(z) = \frac{az+b}{\bar{b}z+\bar{a}}$, with $|a|^2 - |b|^2 = 1$.

A linear fractional map is a function of the form

$$\varphi(z) = \frac{az+b}{cz+d} \quad \text{associated with the matrix } A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \text{ in } \mathfrak{M}_{2\times 2}(\mathbb{C}).$$

Such maps extend to the Riemann sphere $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ by defining $\varphi(-\frac{d}{c}) = \infty$, and $\varphi(\infty) = \frac{a}{c}$. If det$A = 0$, then $\varphi : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is constant. Thus we restrict our attention to the nondegenerate case and use the expressions "linear fractional" and "Möbius" interchangeably. We denote the set of all automorphisms on $\mathbb{C}_\infty$ by $\mathrm{Aut}(\mathbb{C}_\infty)$. The following proposition summarizes some fundamental properties of $\mathrm{Aut}(\mathbb{C}_\infty)$, and eventually details the characteristics of the automorphisms of the upper half-plane, $\mathbb{U}$, denoted by $\mathrm{Aut}(\mathbb{U})$.

Proposition 8

1. $\mathrm{Aut}(\mathbb{C}_\infty) = \left\{\frac{az+b}{cz+d} : a, b, c, d \in \mathbb{C},\ ad - bc \neq 0\right\}$.
2. *The mapping* $\{A \in \mathfrak{M}_{2\times 2}(\mathbb{C}) \mid \det A \neq 0\} \to \mathrm{Aut}(\mathbb{C}_\infty)$ *of* $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \frac{az+b}{cz+d}$ *is a group homomorphism. That is, if* $\varphi_{A_1}(z) = \frac{a_1z+b_1}{c_1z+d_1}$ *and* $\varphi_{A_2}(z) = \frac{a_2z+b_2}{c_2z+d_2}$*, then* $\varphi_{A_1A_2} = \varphi_{A_1} \circ \varphi_{A_2}$.
3. *If* $\varphi \in \mathrm{Aut}(\mathbb{C}_\infty)$*, then* φ *maps circles in* $\mathbb{C}_\infty$ *to circles in* $\mathbb{C}_\infty$ *(circles and lines in* $\mathbb{C}$ *to circles and lines in* $\mathbb{C}$*, respectively).*
4. *If* $\varphi(z) = \frac{az+b}{cz+d} \in \mathrm{Aut}(\mathbb{C}_\infty)$*, then we may choose* $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathfrak{M}_{2\times 2}(\mathbb{C})$ *so that* $\det A = 1$.
5. *If* $\varphi \in \mathrm{Aut}(\mathbb{C}_\infty)$ *and* $\varphi(\mathbb{R}_\infty) = \mathbb{R}_\infty$*, then* $\varphi(z) = \frac{az+b}{cz+d}$ *for some* $a, b, c, d \in \mathbb{R}$.
6. *If* $\varphi \in \mathrm{Aut}(\mathbb{U})$*, then* φ *is a linear fractional map* $\varphi(z) = \varphi_A(z) = \frac{az+b}{cz+d}$*, where* $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *with* $\det A = 1$ *and* $a, b, c, d \in \mathbb{R}$.

For convenience of the reader, we sketch a proof of this proposition.

Proof: (1). Any mapping φ on $\mathbb{C}_\infty$ of the form $\varphi(z) = \frac{az+b}{cz+d}$ with $ad - bc \neq 0$ is invertible with inverse $\varphi^{-1}(z) = \frac{dz-b}{-cz+a}$. Therefore, every such Möbius transformation is an automorphism of $\mathbb{C}_\infty$. Conversely, suppose $\varphi \in \mathrm{Aut}(\mathbb{C}_\infty)$. By composing with a Möbius transformation, we may assume that $\varphi(0) = 0$. Let $f(z) = \varphi(\frac{1}{z})$, $z \in \mathbb{C} \setminus \{0\}$, and write $f(z) = \sum_{n=0}^{\infty} a_n/z^n$ where $a_n = \frac{\varphi^{(n)}(0)}{n!}$.

If f has an essential singularity at 0 (infinitely many $a_n \neq 0$), then $f(\mathbb{D} \setminus \{0\})$ is dense in $\mathbb{C}$ by the Casorati-Weierstrass theorem. At the same time $f(\mathbb{C} \setminus \overline{\mathbb{D}})$ is open and nonempty, and therefore $f(\mathbb{C} \setminus \overline{\mathbb{D}}) \cap f(\mathbb{D} \setminus \{0\}) \neq \emptyset$, contradicting the injectivity of φ. Thus f has a pole at 0, and so $\varphi(z) = \sum_{n=0}^{m} a_n z^n$ is a polynomial. Injectivity of φ then implies that the degree of φ is one, and therefore $\varphi(z) = \varphi(0)z$ for all $z \in \mathbb{C}$. Hence, φ is a M$\ddot{o}$bius transformation as desired.

(2). Let $A_1 = \left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right)$ and $A_2 = \left(\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right)$ be such that $\varphi_{A_1}(z) = \frac{a_1 z + b_1}{c_1 z + d_1}$ and $\varphi_{A_2}(z) = \frac{a_2 z + b_2}{c_2 z + d_2}$. Then

$$\begin{aligned} \left(\varphi_{A_1} \circ \varphi_{A_2}\right)(z) = \varphi_{A_1}(\varphi_{A_2}(z)) &= \frac{a_1 \varphi_{A_2}(z) + b_1}{c_1 \varphi_{A_2} + d_1} \\ &= \frac{(a_1 a_2 + b_1 c_2)z + (a_1 b_2 + b_1 d_2)}{(c_1 a_2 + d_1 c_2)z + (c_1 b_2 + d_1 d_2)}. \end{aligned}$$

On the other hand, $A_1 A_2 = \left(\begin{smallmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{smallmatrix}\right)$, and therefore

$$\varphi_{A_1 A_2}(z) = \frac{(a_1 a_2 + b_1 c_2)z + (a_1 b_2 + b_1 d_2)}{(c_1 a_2 + d_1 c_2)z + (c_1 b_2 + d_1 d_2)} = \varphi_{A_1} \circ \varphi_{A_2}(z).$$

(3). Any circle in C_∞ has the form

$$\alpha(x^2 + y^2) + 2\beta x + 2\gamma y + \delta = 0, \quad (\alpha, \beta, \gamma, \delta \in \mathbb{R}).$$

Alternatively, we can write the above equation as

$$Az\bar{z} + Bz + \bar{B}\bar{z} + C = 0,$$

where $A = \alpha \in \mathbb{R}$, $B = \beta - i\gamma \in \mathbb{C}$, $C = \delta \in \mathbb{R}$. By taking the action of φ on the circle on $\mathbb{C}_\infty$, we have

$$A\varphi(z)\overline{\varphi(z)} + B\varphi(z) + \bar{B}\overline{\varphi(z)} + C = 0,$$

which is equivalent to

$$A^* z\bar{z} + B^* z + \bar{B}^* \bar{z} + C^* = 0, \tag{3.2}$$

where $A^* = A|a|^2 + Ba\bar{c} + \bar{B}\bar{a}c + C|c|^2$, $B^* = Aa\bar{b} + Ba\bar{d} + \bar{B}\bar{b}c + Cc\bar{d}$, $\bar{B}^* = Ab\bar{a} + Bb\bar{c} + \bar{B}\bar{a}d + Cd\bar{c}$, and $C^* = A|b|^2 + Bb\bar{d} + \bar{B}\bar{b}d + C|d|^2$. Clearly, (3.2) is also an equation of a circle in $\mathbb{C}_\infty$.

(4). If $\varphi(z) = \frac{az+b}{cz+d}$ with $\lambda = ad - bc \neq 0$, choose a square root μ of $\frac{1}{\lambda}$, for instance, $\lambda = re^{i\theta} \Rightarrow \mu = \sqrt{r}e^{i\theta/2}$. Then

$$\varphi(z) = \frac{\mu az + \mu b}{\mu cz + \mu d} \quad \text{and } (\mu a)(\mu d) - (\mu b)(\mu c) = \mu^2\lambda = 1.$$

(5). Let $\varphi(z) = \frac{a_1 z + b_1}{c_1 z + d_1}$, $a_1 d_1 - c_1 b_1 \neq 0$.

First, suppose that $c_1 = 0$. Then $d_1 \neq 0$. Put $a = \frac{a_1}{d_1}$, $b = \frac{b_1}{d_1}$, $c = 0$, $d = 1$. Then

$$\varphi(z) = \frac{a_1 z + b_1}{d_1} = az + b = \frac{az+b}{cz+d};$$

where $c = 0 \in \mathbb{R}$, $d = 1 \in \mathbb{R}$, $b = \varphi(0) \in \mathbb{R}$, and $a = \varphi(1) - b \in \mathbb{R}$.

Now, suppose $c \neq 0$. If $x \in \mathbb{R} \setminus \{-d_1/c_1\}$, then $\varphi(x) \in \mathbb{R}$ so that $\varphi(x) = \overline{\varphi(x)}$. That is, $\frac{a_1 x + b_1}{c_1 x + d_1} = \frac{\bar{a}_1 x + \bar{b}_1}{\bar{c}_1 x + \bar{d}_1}$ which implies $a_1\bar{c}_1 = \bar{a}_1 c_1$, $\quad a_1\bar{d}_1 + b_1\bar{c}_1 = \bar{a}_1 d_1 + \bar{b}_1 c_1$, $\quad b_1\bar{d}_1 = \bar{b}_1 d_1$.

Thus, $a_1\bar{c}_1 \in \mathbb{R}$, $a_1\bar{d}_1 + b_1\bar{c}_1 \in \mathbb{R}$, $b_1\bar{d}_1 \in \mathbb{R}$.

Now $\varphi^{-1}(z) = \frac{d_1 z - b_1}{-c_1 z + a_1}$ and $\varphi^{-1}(\mathbb{R}_\infty) = \mathbb{R}_\infty$, then $\bar{a}_1 d_1 + b_1\bar{c}_1 = d_1\bar{a}_1 + (-b_1)(-\bar{c}_1) \in \mathbb{R} \Rightarrow a_1\bar{d}_1 + \bar{b}_1 c_1 \in \mathbb{R}$. Therefore,

$2i\Im(b_1\bar{c}_1) = b_1\bar{c}_1 - \bar{b}_1 c_1 = (a_1\bar{d}_1 + b_1\bar{c}_1) - (a_1\bar{d}_1 + \bar{b}_1 c_1) \in \mathbb{R} \Rightarrow \Im(b_1\bar{c}_1) = 0, \Rightarrow\ b_1\bar{c}_1 \in \mathbb{R}$.

Since $\varphi(\mathbb{R}_\infty) = \mathbb{R}_\infty$, it follows that $a_1\bar{c}_1,\ b_1\bar{c}_1,\ \bar{b}_1 d_1\ \in \mathbb{R}$. Now, put $a = \frac{a_1}{c_1}$, $b = \frac{b_1}{c_1}$, $c = 1$, $d = \frac{d_1}{c_1}$, then

$$\varphi(z) = \frac{a_1 z + b_1}{c_1 z + d_1} = \frac{\frac{a_1}{c_1} z + \frac{b_1}{c_1}}{z + \frac{d_1}{c_1}} = \frac{az + b}{cz + d},$$

where $a = \frac{a_1}{c_1} = \frac{a_1\bar{c}_1}{|c_1|^2} \in \mathbb{R}$, $b = \frac{b_1}{c_1} = \frac{b_1\bar{c}_1}{|c_1|^2} \in \mathbb{R}$, $c = 1 \in \mathbb{R}$, and since $\varphi(\mathbb{R}_\infty) = \mathbb{R}_\infty$, $bd = \frac{d_1\bar{b}_1}{|c_1|^2} \in \mathbb{R}$, $b \in \mathbb{R}$, and thus, $d \in \mathbb{R}$.

(6). Let $\varphi \in \mathrm{Aut}(\mathbb{U})$. Recall that the Cayley transform $\psi(z) = \frac{i(1+z)}{1-z}$ is a linear fractional conformal mapping of $\mathbb{D}$ onto $\mathbb{U}$ with inverse $\psi^{-1}(\omega) = \frac{\omega - i}{\omega + i}$.
Thus, $\varphi \in \mathrm{Aut}(\mathbb{U})$ is equivalent to $\psi^{-1} \circ \varphi \circ \psi \in \mathrm{Aut}(\mathbb{D})$. In particular, $f = \psi^{-1} \circ \varphi \circ \psi$ is a linear fractional transformation by Schwarz's Lemma. This implies that $\psi \circ f \circ \psi^{-1} = \varphi$ is a linear fractional transformation too. Since $\varphi \in \mathrm{Aut}(\mathbb{U})$; $\varphi(\mathbb{U}) = \mathbb{U}$ and therefore $\varphi(\mathbb{R}_\infty) = R_\infty$. Assertions (4) and (5) above then imply that $\varphi(z) = \varphi_A(z) = \frac{az+b}{cz+d}$ with $\det A = 1$, and $a, b, c, d \in \mathbb{R}$. ■

We end this section by briefly considering some facts about the fixed points of automorphisms of the half-plane. We refer to [1] for a detailed exposition. An automorphism of $\mathbb{U}$ extends continuously to $\overline{\mathbb{U}}$ (where the closure is taken in $\mathbb{C}_\infty$), and sends $\overline{\mathbb{U}}$ into itself. As in the case of the disc (see for example [1] or [38]), automorphisms of $\mathbb{U}$ can also be classified as elliptic, parabolic, or hyperbolic depending on the nature of their fixed points. Let $\mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$ be the boundary of $\mathbb{U}$ in $\mathbb{U}_\infty$.

Definition 1

A nontrivial automorphism of the upper half-plane, $\mathbb{U}$*, is said to be elliptic if it has a unique fixed point in* $\mathbb{U}$*, parabolic if it has a unique fixed point on* $\mathbb{R}_\infty$*, hyperbolic if it has two distinct fixed points on* $\mathbb{R}_\infty$*.*

The following is a simple classification of the automorphisms of the upper half-plane, $\mathrm{Aut}(\mathbb{U})$, in terms of the nature and location of their fixed points in regard to the Definition 1 above,

Proposition 9

Let φ *be a nontrivial automorphism of the upper half-plane* $\mathbb{U}$*. Then* φ *is elliptic, respectively parabolic, or hyperbolic, if and only if* $\mathrm{tr}(\varphi) < 2$*, respectively* $= 2$*, or* > 2*, where* $\mathrm{tr}(\varphi)$ *denotes the trace of the* 2×2 *matrix associated with* φ*.*

Proof: By Proposition 8, $\varphi(\omega) = \frac{a\omega+b}{c\omega+d}$ with $ad - bc = 1$, and $\mathrm{tr}(\varphi) = a + d$. Then the fixed point equation of φ is $\varphi(\omega) = \omega$ and is equivalent to

$$c\omega^2 + (d-a)\omega - b = 0. \tag{3.3}$$

If $c \neq 0$, the discriminant $D = (d-a)^2 + 4bc = (a+d)^2 - 4$. Then

φ is elliptic if and only if $(a+d)^2 - 4 < 0$ (equation (3.3) has two distinct complex roots);

φ is parabolic if and only if $(a+d)^2 - 4 = 0$ (equation (3.3) has one repeated real root);

φ is hyperbolic if and only $(a+d)^2 - 4 > 0$ (equation (3.3) has two distinct real roots).

If $c = 0$, then $d = a^{-1}$ and $\varphi(\omega) = a(a\omega+b)$ has a fixed point at ∞. In particular, φ cannot be elliptic. Therefore, it is hyperbolic if and only if it has another fixed point different from

∞. Since $\omega = \frac{b}{a-d}$, this is equivalent to $a \neq d$, that is, $|a+d| > 2$ which completes the proof. ■

3.2 Weighted Composition operators associated with $\mathrm{Aut}(\mathbb{U})$

Recall from Chapter 2 that the Cayley transform $\psi(z) := \frac{i(1+z)}{1-z}$ maps the unit disc $\mathbb{D}$ conformally onto the upper half-plane $\mathbb{U}$ with inverse $\psi^{-1}(\omega) = \frac{\omega-i}{\omega+i}$. Again, we let X denote either the Hardy space $H^p(\mathbb{U})$ or the weighted Bergman space $L^p_a(\mathbb{U},\mu_\alpha)$. We associate with each X, a parameter $\gamma = \frac{\alpha+2}{p}$, $\alpha \geq -1$, where $\alpha = -1$ in case that $X = H^p(\mathbb{U})$. Let $X(\mathbb{D})$ denote the corresponding space of analytic functions on $\mathbb{D}$. We then obtain the following proposition;

Proposition 10

Let $f \in X$, and define $S_\psi f = (\psi')^\gamma f \circ \psi$. Then $S_\psi : X \to X(\mathbb{D})$ is continuous with inverse $S_{\psi^{-1}} g = ((\psi^{-1})')^\gamma g \circ \psi^{-1}$. In fact, if $X = H^p(\mathbb{U})$, then S_ψ is an isometry, and, in the case $X = L^p_a(\mathbb{U},\mu_\alpha)$, $\|S_\psi f\|_{L^p_a(\mathbb{D},m_\alpha)} = 2^{\alpha/p}\|f\|_{L^p_a(\mathbb{U},\mu_\alpha)}$.

Moreover, $S_{\psi^{-1}}$ is an isometry on $H^p(\mathbb{D})$, and if $X(\mathbb{D}) = L^p_a(\mathbb{D},m_\alpha)$, then

$\|S_{\psi^{-1}} g\|_{L^p_a(\mathbb{U},\mu_\alpha)} = 2^{-\alpha/p}\|g\|_{L^p_a(\mathbb{D},m_\alpha)}$.

In particular, $S_\psi^{-1} = S_{\psi^{-1}}$ in the setting of Bergman spaces as well as Hardy spaces.

Proof: First, we suppose that $X = L^p_a(\mathbb{U},\mu_\alpha)$. Let $f \in L^p_a(\mathbb{U},\mu_\alpha)$, then change of variables yields

$$\begin{aligned}\|f\|_{L^p_a(\mu_\alpha)} &= \int_{\mathbb{U}} |f(\omega)|^p (\Im(\omega))^\alpha \, dA(\omega) \\ &= \int_{\mathbb{D}} |f(\psi(z))|^p \, (\Im(\psi(z)))^\alpha \, |\psi'(z)|^2 \, dA(z),\end{aligned}$$

and $\Im(\psi(z)) = \frac{(1-|z|^2)}{2}|\psi'(z)|$. Thus $\|f\|^p_{L^p_a(\mu_\alpha)} = 2^{-\alpha}\|S_\psi f\|^p_{L^p_a(m_\alpha)}$.

For the case $X = H^p(\mathbb{U})$, we may identify $f \in X$ with its boundary values. Then change of variables yields

$$\|f\|^p_{H^p(\mathbb{U})} = \int_{\mathbb{R}} |f(x)|^p \, dx = \int_{\partial\mathbb{D}} |f(\psi(z))|^p |\psi'(z)| \, dm(z)$$
$$= \int_{\partial\mathbb{D}} |(\psi'(z))^\gamma (f \circ \psi)(z)|^p \, dm(z),$$

where $dm(e^{i\theta}) = d\theta$ denotes arc-length measure on $\partial\mathbb{D}$. Thus $\|S_\psi f\|_{H^p(\mathbb{D})} = \|f\|_{H^p(\mathbb{U})}$.

Similarly, if $g \in L^p_a(\mathbb{D}, m_\alpha)$, then again by change of variables, we obtain

$$\|g\|^p_{L^p_a(m_\alpha)} = \int_{\mathbb{D}} |g(z)|^p (1 - |z|^2)^\alpha \, dA(z)$$
$$= \int_{\mathbb{U}} |g(\psi^{-1}(\omega))|(1 - |\psi^{-1}(\omega)|^2)^\alpha |(\psi^{-1})'|^2 \, dA(\omega),$$

where $(1 - |\psi^{-1}(\omega)|^2)) = 2|(\psi^{-1})'(\omega)|\Im(\omega)$. Thus

$$\|g\|^p_{L^p_a(m_\alpha)} = 2^\alpha \int_{\mathbb{U}} |(\psi^{-1})'|^{\alpha+2} |g \circ \psi^{-1}|^p (\Im(\omega))^\alpha \, dA(\omega) = 2^\alpha \|S_{\psi^{-1}} g\|^p_{L^p_a(\mu_\alpha)}.$$

If $g \in H^p(\mathbb{D})$, then

$$\|g^p\|_{H^p(\mathbb{D})} = \int_{\partial\mathbb{D}} |g(z)|^p \, dm(z) = \int_{\mathbb{R}} |g(\psi^{-1}(x))|^p |(\psi^{-1})'| \, dx$$
$$= \|S_{\psi^{-1}} g\|^p_{H^p(\mathbb{R})}.$$

■

More generally, let $\{V_1, V_2\} = \{\mathbb{D}, \mathbb{U}\}$, and let $LF(V_i, V_j)$ denote the collection of conformal mappings from V_i onto V_j. Then $LF(V_i, V_i) = \mathrm{Aut}(V_i)$, and if $h \in LF(V_i, V_j)$, then $g \in \mathrm{Aut}(V_j) \mapsto h^{-1} \circ g \circ h \in \mathrm{Aut}(V_i)$ is an isomorphism from $\mathrm{Aut}(V_i)$ onto $\mathrm{Aut}(V_j)$. The following is a generalized form of the definition in Proposition 10,

Definition 2

For each $g \in LF(V_i, V_j)$, we define a weighted composition operator $S_g : \mathcal{H}(V_j) \to \mathcal{H}(V_i)$, by

$$S_g f(z) = (g'(z))^\gamma f(g(z)), \quad \text{for all } z \in V_i. \tag{3.4}$$

We note that if $g \in LF(V_i, V_j)$ and $h \in LF(V_j, V_i)$, then it is clear by chain rule that $S_h S_g = S_{gh}$ and $S_g^{-1} = S_{g^{-1}}$. Indeed,

$$S_h S_g f(z) = S_h\left((g')^\gamma f(g(z))\right) = (h')^\gamma \left(g'(h)\right)^\gamma f\left(g(h(z))\right)$$

$$= \left((g \circ h)'(z)\right)^\gamma f\left(g \circ h\right)(z) = S_{g\circ h} f(z). \quad \text{In particular } S_{g^{-1}} = S_g^{-1}.$$

In the next Proposition, it turns out that the weighted composition operators S_g are invertible isometries on both the Hardy and weighted Bergman spaces.

Proposition 11

1. *If $g \in \mathrm{Aut}(\mathbb{U})$, then for every p, $1 \le p < \infty$ and $\alpha > -1$, the operator S_g is a surjective isometry on $L^p_a(\mathbb{U}, \mu_\alpha)$.*
2. *If $g \in \mathrm{Aut}(\mathbb{U})$, then for every p, $1 \le p < \infty$, the operator S_g is a surjective isometry on $H^p(\mathbb{U})$.*
3. *If $g \in LF(\mathbb{U}, \mathbb{D})$, then S_g is a surjective isometry from $H^p(\mathbb{D})$ onto $H^p(\mathbb{U})$.*
4. *If $g \in LF(\mathbb{U}, \mathbb{D})$, then $2^{\alpha/p} S_g$ is a surjective isometry from $L^p_a(\mathbb{D}, m_\alpha)$ onto $L^p_a(\mathbb{U}, mu_\alpha)$ with inverse $2^{-\alpha/p} S_{g^{-1}}$.*

Proof: For (1), we recall from Section 3.1 that every automorphism g of $\mathbb{U}$ is a linear fractional map of the form $g(z) = \frac{az+b}{cz+d}$ corresponding to a real 2×2 matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $\det(A) = 1$. In fact, this representation is unique up to multiples of ± 1. If g and

A are as above, then a computation yields that $|g'(z)|\Im(z) = \Im(g(z))$, and therefore, for every $f \in L^p_a(\mathbb{U}, \mu_\alpha)$,

$$\begin{aligned}\|S_g f\|^p &= \int_{\mathbb{U}} |f(g(z))|^p (|g'(z)|\Im(z))^\alpha |g'(z)|^2 \, dA(z) \\ &= \int_{\mathbb{U}} |f(\omega)|^p (\Im(\omega))^\alpha \, dA(\omega) = \|f\|^p.\end{aligned}$$

Thus S_g is an isometry on $L^p_a(\mathbb{U}, \mu_\alpha)$, and $S_g S_{g^{-1}} = I$ by the remarks preceding the proposition.

For (2), it is clear that $g \in \mathrm{Aut}(\mathbb{U})$ if and only if $h = \psi^{-1} \circ g \circ \psi \in \mathrm{Aut}(\mathbb{D})$, ψ is the Cayley transform, and in this case, S_h is a surjective isometry on $H^p(\mathbb{D})$. By Proposition 10, S_ψ is an isometry from $H^p(\mathbb{U})$ onto $H^p(\mathbb{D})$, and thus the desired result follows.

Now, if $g \in LF(\mathbb{U}, \mathbb{D})$, then $h = \psi \circ g \in \mathrm{Aut}(\mathbb{U})$ and so $S_h = S_g \circ S_\psi$ is a surjective isometry on $H^p(\mathbb{U})$. Thus $S_g = S_h S_{\psi^{-1}}$ is a surjective isometry from $H^p(\mathbb{D})$ onto $H^p(\mathbb{U})$ which proves (3).

Finally, if g is a conformal map from $\mathbb{U}$ onto $\mathbb{D}$, then $h = \psi \circ g \in \mathrm{Aut}(\mathbb{U})$ and $S_g = S_h S_{\psi^{-1}}$. S_h is a surjective isometry by assertion (1), and by the proposition 10, $2^{-\alpha/p} S_{\psi^{-1}}$ is an isometry from $L^p_a(\mathbb{D}, m_\alpha)$ onto $L^p_a(\mathbb{U}, \mu_\alpha)$. Thus $2^{-\alpha/p} S_g$ is also a surjective isometry as desired. This completes the proof. ∎

For the sake of completeness, we state the following two Theorems due to M. Abate; for details see [1, Theorems 1.4.22 and 1.4.23].

Theorem 8

Every fixed - point free one-parameter semigroup $\Phi : \mathbb{R}^+ \to \mathcal{H}(\mathbb{D})$ *is of the form*

$$\Phi_t(z) = g^{-1}(g(z) + it),$$

where g is a biholomorphism between $\mathbb{D}$ and a domain $\mathcal{D} \subset \mathbb{C}$ vertically invariant; g is uniquely determined up to an additive constant. Furthermore, Φ is a one-parameter group if and only if $\mathcal{D}$ is either a vertical strip or a vertical half-plane.

Theorem 9

Every nontrivial one-parameter semigroup $\Phi : \mathbb{R}^+ \to \mathcal{H}(\mathbb{D})$ with fixed point $\tau \in \mathbb{D}$ is of the form

$$\Phi_t(z) = g^{-1}(e^{-\lambda t} g(z))$$

where $\lambda \in \mathbb{C} \setminus \{0\}$ with $\Re(\lambda) \geq 0$, and g is a biholomorphism between $\mathbb{D}$ and λ-invariant domain $\mathcal{D} \subset \mathbb{C}$ so that $g(\tau) = 0$; λ is uniquely determined, and g is uniquely determined up to a multiplicative constant. Furthermore, Φ is a one-parameter group if and only if $\Re(\lambda) = 0$ and $\mathcal{D}$ is a disc.

The next result is an immediate consequence of Theorem 8 and Theorem 9 above. It is in this theorem that we characterize all the automorphisms of the upper half-plane. This is a fundamental result in this study.

Theorem 10

Let $\varphi : \mathbb{R} \to \mathrm{Aut}(\mathbb{U})$ be a nontrivial continuous group homomorphism. Then exactly one of the following cases holds:

1. *There exists $k > 0$, $k \neq 1$, and $g \in \mathrm{Aut}(\mathbb{U})$ so that $\varphi_t(z) = g^{-1}(k^t g(z))$ for all $z \in \mathbb{U}$ and $t \in \mathbb{R}$.*
2. *There exists $k \in \mathbb{R}$, $k \neq 0$, and $g \in \mathrm{Aut}(\mathbb{U})$ so that $\varphi_t(z) = g^{-1}(g(z) + kt)$ for all $z \in \mathbb{U}$ and $t \in \mathbb{R}$.*
3. *There exists $k \in \mathbb{R}$, $k \neq 0$, and a conformal mapping g of $\mathbb{U}$ onto $\mathbb{D}$ such that $\varphi_t(z) = g^{-1}(e^{ikt} g(z))$ for all $z \in \mathbb{U}$ and $t \in \mathbb{R}$. Equivalently, there exist $\theta \in \mathbb{R} \setminus \{0\}$ and $h \in \mathrm{Aut}(\mathbb{U})$ so that*

$$\varphi_t(z) = h^{-1}\left(\frac{h(z)\cos(\theta t) - \sin(\theta t)}{h(z)\sin(\theta t) + \cos(\theta t)}\right).$$

Proof: Let $\varphi : \mathbb{R} \to \mathrm{Aut}(\mathbb{U})$ be a nontrivial continuous group homomorphism. Then, by Theorems 8 and 9 above, exactly one of the following cases holds:

(a) There is a conformal map h of $\mathbb{U}$ onto a horizontal strip $V = \{z : c < \Im(z) < d\}$ such that $\varphi_t(z) = h^{-1}(h(z) + t)$ for all $z \in \mathbb{U}$, $t \in \mathbb{R}$; moreover, h is unique up to an additive constant.

(b) There is a conformal map h of $\mathbb{U}$ onto a horizontal half plane $W = \{z : c < \Im(z)\}$ or $W = \{z : \Im(z) < c\}$ such that $\varphi_t(z) = h^{-1}(h(z) + t)$for all $z \in \mathbb{U}$, $t \in \mathbb{R}$; moreover, h is unique up to an additive constant.

(c) There is a unique real number $k \neq 0$ and a conformal map h of $\mathbb{U}$ onto a disc $D = \{z : |z| < R\}$ such that $\varphi_t(z) = h^{-1}(e^{ikt}h(z))$ for all $z \in \mathbb{U}$, $t \in \mathbb{R}$. The mapping h is unique up to an multiplicative constant.

If $k > 0$, $k \neq 1$, then the automorphism group $\varphi_t(z) = k^t z$ has the form (a) corresponding to $h(z) = \ln(z)/\ln(k)$. Conversely, for φ and h as in (a), we may, by adding a constant to the conformal mapping h, assume that $V = \{z : 0 < \Im(z) < c\}$ for some $c > 0$. Let $u(z) = \frac{c}{\pi}\ln(z)$. Then $g = u^{-1} \circ h \in \mathrm{Aut}(\mathbb{U})$, and $g \circ \varphi_t \circ g^{-1}(z) = e^{\pi/c\, t}z$. Thus $\varphi_t(z) = g^{-1}(k^t g(z))$ for $k = e^{\pi/c}$.

For φ and h as in (b), we may again add a constant to h to obtain $\mathcal{R}(h) = \mathbb{U}$, in which case we are done, or $\mathcal{R}(h) = -\mathbb{U}$. In the latter case, taking $g = -h$ yields $g \circ \varphi_t \circ g^{-1}(z) = z - t$, or $\varphi_t(z) = g^{-1}(g(z) + bt)$ with $b = -1$. The first statement in (3) follows from case (c) above by re-scaling. For the second part of (3), suppose that $\varphi_t(z) = g^{-1}(e^{ikt}g(z))$

for some $k \in \mathbb{R} \setminus \{0\}$ and $g \in LF(\mathbb{U}, \mathbb{D})$. Then $h := \psi \circ g \in \mathrm{Aut}(\mathbb{U})$ and by a simple calculation we obtain

$$\psi(e^{ikt}\psi^{-1}(\omega)) = \frac{\omega \cos\left(-\frac{k}{2}t\right) - \sin\left(-\frac{k}{2}t\right)}{\omega \sin\left(-\frac{k}{2}t\right) + \cos\left(-\frac{k}{2}t\right)}.$$

The second statement of (3) follows. ■

Since every continuous one-parameter semigroup $\varphi : \mathbb{R}^+ \to \mathrm{Aut}(\mathbb{U})$ extends uniquely to a continuous one-parameter group via $\varphi(t) = \varphi^{-1}(-t)$ for $t < 0$, continuous one-parameter semigroups of automorphisms are also of three types.

We illustrate the above Theorem 10 with the aid of the following examples;

Example 1 *This example corresponds to case 1 of Theorem 10. Let* $g(z) = \frac{z}{1+z}$. *Then a straightforward calculation shows that*

$$\Im(g(z)) \;=\; \frac{\Im(z)}{|1+z|^2} \;>\; 0 \;\;\Rightarrow\;\; g(\mathbb{U}) \;\subseteq\; \mathbb{U}.$$

Clearly, $g \in \mathrm{Aut}(\mathbb{U})$ *with the inverse* $g^{-1}(z) = \frac{z}{1-z}$. *Taking* $k = e^{-1}$, *we get*

$$\varphi_t(z) = g^{-1}\left(e^{-t}g(z)\right) = \frac{e^{-t}\frac{z}{1+z}}{1 - e^{-t}\frac{z}{1+z}} = \frac{e^{-t}z}{(1-e^{-t})z+1},$$

and the weighted composition operator associated with it is given by

$$S_{\varphi_t}f(z) = \frac{e^{-t\gamma}}{((1-e^{-1})z+1)^{2\gamma}} f\left(\frac{e^{-t}z}{(1-e^{-t})z+1}\right).$$

In the case that we take g *to be the identity, then* $\varphi_t(z) = e^{-t}z$ *and the weighted composition operator associated with it is,* $S_{\varphi_t}f(z) = e^{-t\gamma}f(e^{-t}z)$. *This is the group that was considered by Siskakis and Arvanitidis in [4] in their study of Cesàro operator on* $H^p(\mathbb{U})$. *We shall study this group in detail in the next chapter.* ■

Example 2 *This example corresponds to case 2 of Theorem 10. Let* $g(z) = \frac{z}{1-z}$ *so that its inverse is given by* $g^{-1}(z) = \frac{z}{1+z}$*. Now, taking* $k = 1$*, we have*

$$\varphi_t(z) = g^{-1}(g(z) + t) = \frac{\frac{z}{1-z} + t}{1 - \frac{z}{1-z} - t} = \frac{(1-t)z + t}{-tz + 1 + t},$$

and the weighted composition operator associated with it is,

$$S_{\varphi_t} f(z) = \frac{1}{(-tz + 1 + t)^{2\gamma}} f\left(\frac{(1-t)z + t}{-tz + 1 + t}\right).$$

We shall consider the analysis of this semigroup in Chapter 5. ■

Example 3 *This last example corresponds to the case 3 of Theorem 10. Here, we consider* $g : \mathbb{U} \to \mathbb{D}$ *conformal and the readily available option is the map* ψ *with inverse* ψ^{-1} *already defined in section 3.1. Therefore, we take* $g(z) = \psi^{-1}(z) = \frac{z-i}{z+i}$ *with* $g^{-1}(z) = \psi(z) = \frac{i(1+z)}{1-z}$*. Also, let* $k = -2$ *and a direct computation yields*

$$1 + e^{i\theta} = 2\cos\left(\frac{\theta}{2}\right) e^{\frac{i\theta}{2}} \quad \textit{and} \quad 1 - e^{i\theta} = -2i\sin\left(\frac{\theta}{2}\right) e^{\frac{i\theta}{2}}. \tag{3.5}$$

Then

$$\begin{aligned}
\varphi_t(z) &= g^{-1}(e^{-2it}g(z)) = \frac{i\left(1 + e^{-2it}\frac{z-i}{z+i}\right)}{1 - e^{-2it}\frac{z-i}{z+i}} \\
&= \frac{i\left((1 + e^{-2it})z + i(1 - e^{-2it})\right)}{(1 - e^{-2it})z + i(1 + e^{-2it})} \\
&= \frac{i\left((2\cos(-t)e^{-it})z + i(-2i\sin(-t)e^{-it})\right)}{(-2i\sin(-t)e^{-it})z + i(2\cos(-t)e^{-it})} \quad \left(\textit{ by taking } \theta = -2t \textit{ in } (3.5)\right), \\
&= \frac{z\cos t - \sin t}{z\sin t + \cos t},
\end{aligned}$$

and the corresponding weighted composition operator is;

$$S_{\varphi_t} f(z) = \frac{1}{(z\sin t + \cos t)^{2\gamma}} f\left(\frac{z\cos t - \sin t}{z\sin t + \cos t}\right).$$

Further analysis of this group will be considered in Chapter 5. ■

Combining Theorem 10 and the Proposition 11, we obtain three similarity classes of invertible isometries associated with continuous one-parameter groups of automorphisms of the half-plane. We shall focus on the following three groups φ_t of automorphisms of the upper half plane $\mathbb{U}$ and their corresponding weighted composition operators, which of course, are the groups of invertible isometries $T_t f := S_{\varphi_t} f = (\varphi_t')^\gamma f(\varphi_t)$ on the spaces of analytic functions $X = H^p$ or L^p_a, $1 \leq p < \infty$.

1. **The Scaling group**
   The group of automorphisms $\varphi_t(z) = k^t z$ with $k > 0$, $k \neq 1$, $t \in \mathbb{R}$ and the corresponding group of isometries $T_t = S_{\varphi_t}$.

2. **The Translation group**
   The group of automorphisms $\varphi_t(z) = z + kt$ with $t,\ k \in \mathbb{R}$, $k \neq 0$ and the associated group of isometries $T_t = S_{\varphi_t}$.

3. **The Rotation group**
   The group of automorphisms $\varphi_t(z) = e^{ikt} z$, $k \in \mathbb{R}$, $k \neq 0$, but instead with isometries on the spaces $X(\mathbb{D})$ of the form $T_t f(z) = S_{\varphi_t} f(z) := e^{ict} f(e^{ikt} z)$ with $c \in \mathbb{R}$.

In fact in terms of the nature and location of the fixed points, the scaling, the translation, and the rotation groups can be classified exactly as hyperbolic, parabolic and elliptic respectively. Indeed, following Proposition 8, $\varphi_t \in \mathrm{Aut}(\mathbb{U})$ is of the form $\varphi_t(z) = \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{R}$ with $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $\det(A) = 1$.

For the translation group, $\varphi_t(z) = z + kt,\ k \neq 0$, implying that $A = \left(\begin{smallmatrix} 1 & kt \\ 0 & 1 \end{smallmatrix}\right)$ and $\det(A) = 1$. Therefore, $\mathrm{tr}(A) = 2$ and by Proposition 9, this group is parabolic.

For the rotation group, $\varphi_t(z) = e^{ikt} z$, $k \neq 0$, implying that $\bar{A} = \begin{pmatrix} \sqrt{e^{ikt}} & 0 \\ 0 & \frac{1}{\sqrt{e^{ikt}}} \end{pmatrix}$ so that $\det(\bar{A}) = 1$. Now $\mathrm{tr}(\bar{A}) = \frac{1}{e^{(ikt)/2}} \left(e^{ikt} + 1 \right) < 2$, and again, by Proposition 9, this group is elliptic.

Finally, for the scaling group, $\varphi_t(z) = k^t z,\ k > 0,\ k \neq 1$, implying that $\bar{A} = \begin{pmatrix} \sqrt{k^t} & 0 \\ 0 & \frac{1}{\sqrt{k^t}} \end{pmatrix}$ so that $\det(\bar{A}) = 1$. Now, $\operatorname{tr}(\bar{A}) = \sqrt{k^t} + \frac{1}{\sqrt{k^t}} > 2$ and thus, this group is hyperbolic by Proposition 9.

We have so far obtained three groups of invertible isometries associated with one-parameter groups of automorphisms of the half-plane. The next result which is the last in this chapter shows that the group $(T_t)_{t\in\mathbb{R}}$ has an additional property of strong continuity. The strong continuity property has far reaching consequences in semigroup theory. In particular, the Hille - Yosida theorem (Theorem 2) gives a necessary and sufficient condition for an operator to generate a strongly continuous one-parameter semigroup in terms of norms of its resolvents.

Theorem 11

For $X = H^p(\mathbb{U})$ *or* $L^p_a(\mathbb{U}, \mu_\alpha)$, $1 \leq p < \infty$, *let* $\varphi : \mathbb{R} \to \operatorname{Aut}(\mathbb{U})$ *be a nontrivial continuous group homomorphism, and let* $T_t = S_{\varphi_t}$ *be the associated isometry. Then* $(T_t)_{t\in\mathbb{R}}$ *is strongly continuous in* $\mathcal{L}(X)$.

Proof: First consider the case that $X = L^p_a(\mathbb{U}, \mu_\alpha)$, $1 \leq p < \infty$ and $\alpha > -1$. Let $f \in L^p_a(\mathbb{U}, \mu_\alpha)$ and suppose that $t_n \to 0$ in $\mathbb{R}$. Let $f_n = T_{t_n} f$. Then considering each of the three types of $\operatorname{Aut}(\mathbb{U})$ given by Theorem 10; $f_n(z) \to f(z)$ for each $z \in \mathbb{U}$, and $\|f_n\|_p = \|f\|_p$ for each n. If $g_n := 2^{p-1}(|f|^p + |f_n|^p) - |f - f_n|^p$, then $g_n \geq 0$ and $g_n(z) \to 2^p |f(z)|^p$ on $\mathbb{U}$. It now follows from Fatou's lemma that

$$\limsup_{n\to\infty} \int_{\mathbb{U}} |f - f_n|^p \, d\mu_\alpha = 0,$$

and therefore $\|T_{t_n} f - f\|_{L^p_a(\mu_\alpha)} \to 0$ as $n \to \infty$.

For the Hardy space case, since each T_t is an isometry, it suffices to show that $\|T_t f - f\|_{H^p(\mathbb{U})} \to 0$ as $t \to 0$ on the dense subset $\{f \in H^p : f \text{ is continuous on } \overline{\mathbb{U}}\}$. In each case of Theorem 10, $\varphi_t'(x) \to 1$ and $\varphi_t(x) \to x$ for all $x \in \mathbb{R} \setminus \{f^{-1}(\infty), f(\infty)\}$, and therefore, for $f \in H^p$ such that f is continuous on $\overline{\mathbb{U}}$, $T_t f(x) \to x$ for almost all $x \in \mathbb{R}$ as $t \to 0$. We now apply Fatou's lemma as above to conclude that

$$\int_{-\infty}^{\infty} |f(x) - T_t f(x)|^p \, dx \to 0 \quad \text{as } t \to 0.$$

■

In Chapters 4 and 5, we shall study each of the above three groups of isometries $(T_t)_{t \in \mathbb{R}}$ associated with the one-parameter groups of automorphisms $(\varphi_t)_{t \in \mathbb{R}}$ of the half-plane. We shall determine the infinitesimal generator of each of the three groups $(T_t)_{t \in \mathbb{R}}$, then proceed to study their spectral properties and consequently obtain the resolvents of the generators and their properties as well.

CHAPTER 4

SCALING GROUP AND CESÀRO-LIKE OPERATORS

In this chapter, we analyze the scaling group. We shall obtain the generator of this group and determine its spectral properties. Using both spectral and semigroup theory, we obtain the resolvents of the generator concretely as integral operators which are exactly the Cesàro-like operators on analytic spaces of the half-plane. Based on the duality properties of Hardy and Bergman spaces of the half-plane established in Chapter 2, we determine the adjoints of these integral operators in Section 4.2 of this chapter.

4.1 The Scaling group

As noted in Chapter 3, for the scaling group, we consider $\varphi_t(z) = k^t z$, $k > 0,\ k \neq 1$, and $t \in \mathbb{R},\ z \in \mathbb{U}$. Then the corresponding group of isometries is given by $T_t f(z) = (\varphi_t'(z))^\gamma f(\varphi_t(z)) = k^{t\gamma} f(k^t z)$, for every $f \in X$. Let Γ_k be its infinitesimal generator. Unless otherwise specified, we shall let X denote either the Hardy spaces $H^p(\mathbb{U})$ or the weighted Bergman spaces $L^p_a(\mathbb{U}, \mu_\alpha)$. We associate with each X a parameter $\gamma := \frac{\alpha+2}{p}$, where $\alpha = 1$ for $H^p(\mathbb{U})$ and $\alpha > -1$ for $L^p_a(\mathbb{U}, \mu_\alpha)$. Also, we let $X(\mathbb{D})$ denotes the corresponding spaces on the unit disc $\mathbb{D}$. We obtain the generator of the group $(T_t)_{t\in\mathbb{R}}$ in the following result,

Proposition 12

The infinitesimal generator Γ_k *of* $(T_t)_{t\in\mathbb{R}} \subset \mathcal{L}(X)$ *is given by* $\Gamma_k(f)(z) = \ln k(\gamma f(z) + zf^{'}(z))$ *with domain* $\mathcal{D}(\Gamma_k) = \{f \in X : zf^{'}(z) \in X\}$.

Proof: Let $f \in \mathcal{D}(\Gamma_k)$ in X, then the growth conditions given by equations (2.1) and (2.2) imply that for all $z \in \mathbb{U}$,

$$\begin{aligned}\Gamma_k f(z) &= \lim_{t\to 0^+} \frac{k^{t\gamma} f(k^t z) - f(z)}{t} = \frac{\partial}{\partial t}\left(k^{t\gamma} f(k^t z)\right)\Big|_{t=0} \\ &= \left[(\gamma \ln k)k^{t\gamma} f(k^t z) + (\ln k)k^t z k^{t\gamma} f'(k^t z)\right]\Big|_{t=0} \\ &= \ln k(\gamma f(z) + zf^{'}(z)).\end{aligned}$$

Therefore, $\mathcal{D}(\Gamma_k) \subset \{f \in X : zf^{'}(z) \in X\}$. Conversely let $f \in X$ be such that $zf^{'}(z) \in X$. Then for $z \in \mathbb{U}$ we have

$$\begin{aligned}T_t(f)(z) - f(z) &= \int_0^t \frac{\partial}{\partial s}\left(k^{\gamma s} f(\varphi_s(z))\right) ds \\ &= \int_0^t \left[(\gamma \ln k)k^{s\gamma} f(\varphi_s(z)) + k^{s\gamma}(\ln k)\varphi_s(z) f'(\varphi_s(z))\right] ds \\ &= \int_0^t k^{s\gamma} \ln k \left[\gamma f(\varphi_s(z)) + \varphi_s(z) f'(\varphi_s(z))\right] ds \\ &= \int_0^t T_s F(z) ds, \quad \text{where } F(z) = \ln k(\gamma f(z) + zf^{'}(z)).\end{aligned}$$

Thus,

$$\lim_{t\to 0^+} \frac{T_t(f) - f}{t} = \lim_{t\to 0^+} \frac{1}{t}\int_0^t T_s F\, ds,$$

and strong continuity of $(T_s)_{s\geq 0}$ implies that $\frac{1}{t}\int_0^t \|T_s F - F\|\, ds \to 0$ as $t \to 0^+$. Thus $D(\Gamma_k) \supseteq \{f \in X : zf^{'}(z) \in X\}$ which completes the proof. ∎

If $k = e^{-1}$, then the group $(T_t)_{t\in\mathbb{R}}$ becomes $T_t f(z) = e^{-t\gamma} f(e^{-t} z)$. Let Γ denotes its generator. Then we obtain the following result,

Proposition 13

Let X denotes one of the spaces $H^p(\mathbb{U})$ or $L^p_a(\mathbb{U},\mu_\alpha)$, $1 \le p < \infty$. Then

1. $\Gamma_k = \ln k\,\Gamma$ *with domain* $\mathcal{D}(\Gamma) = \mathcal{D}(\Gamma_k) = \{f \in X : zf'(z) \in X\}$.
2. $\sigma(\Gamma_k) = \ln k\,\sigma(\Gamma)$ *and* $\sigma_p(\Gamma_k) = \ln k\,\sigma_p(\Gamma)$.

If fact, $\lambda \in \rho(\Gamma)$ *if and only if* $\ln k\,\lambda \in \rho(\Gamma_k)$, *and*

$$R(\ln k\,\lambda, \Gamma_k) = \frac{1}{\ln k} R(\lambda, \Gamma). \tag{4.1}$$

Proof: The first claim follows immediately from Proposition 12. For the second claim, suppose $\lambda \in \rho(\Gamma)$, then

$$(\ln k\,\lambda - \Gamma_k)\,\frac{1}{\ln k} R(\lambda,\Gamma) = (\ln k\,\lambda - \ln k\,\Gamma)\,\frac{1}{\ln k} R(\lambda,\Gamma) = \frac{\ln k}{\ln k}(\lambda - \Gamma)R(\lambda,\Gamma) = I,$$

and if $f \in \mathcal{D}(\Gamma)$, then

$$\frac{1}{\ln k} R(\lambda,\Gamma)\,(\ln k\,\lambda - \Gamma_k)\,f = \frac{\ln k}{\ln k} R(\lambda,\Gamma)(\lambda - \Gamma)f = f.$$

Conversely, if $\ln k\,\lambda \in \rho(\Gamma_k)$, then

$$(\lambda - \Gamma)\ln k\,R(\ln k\,\lambda, \Gamma_k) = (\ln k\,\lambda - \ln k\Gamma)\,R(\ln k\,\lambda,\Gamma_k) = I,$$

and if $f \in \mathcal{D}(\Gamma_k)$, then

$$\ln k\,R(\ln k\,\lambda,\Gamma_k)(\lambda - \Gamma)f = R(\ln k\,\lambda,\Gamma_k)f = R(\ln k\,\lambda,\Gamma_k)(\ln k\,\lambda - \ln k\,\Gamma)f = f,$$

which completes the proof. ■

As a result of Proposition 13, from now on, we restrict our attention to the case $k = e^{-1}$. That is, for every $f \in X$, we consider the group $T_t f(z) = e^{-t\gamma} f(e^{-t}z)$ and its generator as $\Gamma f(z) = -\gamma f(z) - zf'(z)$. The next result details some spectral properties of the infinitesimal generator Γ of this group.

Proposition 14

Let X denote one of the spaces $H^p(\mathbb{U})$ or $L^p_a(\mathbb{U}, \mu_\alpha)$, $1 \leq p < \infty$. Then $\sigma_p(\Gamma, X) = \emptyset$ and $\sigma(\Gamma, X) = i\mathbb{R}$. In particular Γ is an unbounded operator.

Proof: Let $\lambda \in \mathbb{C}$. Then $\lambda \in \sigma_p(\Gamma)$ if and only if $\Gamma f(z) = \lambda f(z)$ for some $0 \neq f \in X$. This is equivalent to the differential equation

$$-zf^{'}(z) - (\lambda + \gamma) f(z) = 0,$$

whose general solution is $f(z) = cz^{-(\gamma+\lambda)}$. It follows from lemma 3, that $f = 0$ and therefore $\sigma_p(\Gamma, X) = \emptyset$.

Since each T_t is an invertible isometry, its spectrum satisfies $\sigma(T_t) \subseteq \partial\mathbb{D}$, and the spectral mapping theorem for strongly continuous groups, Theorem 3, implies that $e^{t\sigma(\Gamma)} \subseteq \sigma(T_t)$. Thus, $e^{t\sigma(\Gamma)} \subseteq \partial\mathbb{D} \Rightarrow |e^{t\sigma(\Gamma)}| = 1 \Rightarrow e^{t\Re(\omega)} = 1 \Rightarrow \Re(\omega) = 0$ for $\omega \in \sigma(\Gamma)$. It immediately follows that $\sigma(\Gamma, X) \subseteq i\mathbb{R}$.

We now need to show the reverse inclusion: $i\mathbb{R} \subseteq \sigma(\Gamma, X)$. Fix $\lambda \in i\mathbb{R}$ and assume $\lambda \notin \sigma(\Gamma, X)$ which implies that the resolvent operator $R(\lambda, \Gamma) : X \to X$ is bounded. Consider the function $h(\omega) = i(\lambda + \gamma)(\omega + i)^{-(\gamma+\lambda+1)}$. Then $\Re(-(\gamma + \lambda + 1)) = -\gamma - 1 < -\gamma$, and therefore by Lemma 3, $h \in X$. The image function $f = R(\lambda, \Gamma)h$ is equivalent to $(\lambda - \Gamma)f = h$ which yields the differential equation

$$f'(\omega) + (-\gamma + \lambda)\,\frac{1}{\omega} f(\omega) = -\frac{h(\omega)}{\omega},$$

whose general solution is

$$f(\omega) = (i + \omega)^{-(\gamma+\lambda)} + c\omega^{-(\gamma+\lambda)},$$

for some constant c. From Proposition 10, $f \in X$ if and only if $S_\psi f \in X(\mathbb{D})$. Now we compute $S_\psi f$ as follows; $i + \psi(\omega) = \frac{2i}{1-\omega}$, $\psi'(\omega) = \frac{2i}{(1-\omega)^2}$ and

$$\begin{aligned} S_\psi f(\omega) &= (\psi'(\omega))^\gamma (f \circ \psi)(\omega) \\ &= \left(\frac{2i}{(1-\omega)^2}\right)^\gamma \left[\left(\frac{2i}{1-\omega}\right)^{-(\gamma+\lambda)} + c\left(\frac{i(1+\omega)}{1-\omega}\right)^{-(\gamma+\lambda)}\right] \\ &= \left(c_1 + c_2 c(1+\omega)^{-(\gamma+\lambda)}\right)\left((1-\omega)^{-(\gamma-\lambda)}\right), \end{aligned}$$

where c_1, c_2 are nonzero constants. Since $\Re(-(\gamma+\lambda)) = -(\gamma-\lambda) = -\gamma$, $S_\psi f \notin X(\mathbb{D})$ for any value of c by Lemma 3, and consequently, $f \notin X$ for any c. Thus $h \notin \mathcal{R}(\lambda - \Gamma)$, and so $\sigma(\Gamma, X) = i\mathbb{R}$. ■

We can now establish the resolvents of the infinitesimal generator Γ of the group $T_t f(z) = e^{-t\gamma} f(e^{-t}z)$. Specifically, we obtain concrete representation of the resolvents of the generator Γ as integral operators which, interestingly, are exactly the Cesàro-like operators on the half-plane, $\mathcal{C}_\nu$.

Theorem 12

Let X denote one of the spaces $H^p(\mathbb{U})$ or $L^p_a(\mathbb{U}, \mu_\alpha)$, with $\alpha > -1$ and $1 \le p < \infty$, ($\alpha = -1$ in the Hardy space case), and let $\gamma = (\alpha+2)/p$. For $\nu \in \mathbb{C}$, the resolvent operator $R(\nu - \gamma, \Gamma)$ on X is given concretely as integral operator as follows:

1. *If $\Re(\nu) > \gamma$, then*

$$R(\nu - \gamma, \Gamma)h(z) = \frac{1}{z^\nu}\int_0^z \omega^{\nu-1} h(\omega)\, d\omega \ := \mathcal{C}_\nu h(z).$$

2. *If $\Re(\nu) < \gamma$, then*

$$R(\nu - \gamma, \Gamma)h(z) = -\frac{1}{z^\nu}\int_z^\infty \omega^{\nu-1} h(\omega)\, d\omega \ := \mathcal{C}_\nu h(z).$$

Moreover,

(a) $\sigma(\mathcal{C}_\nu, X) = \left\{ w \in \mathbb{C} : |w - \frac{1}{2\Re(\nu-\gamma)}| = \frac{1}{2|\Re(\nu-\gamma)|} \right\}$, *and*

(b) $\mathcal{C}_\nu$ *is a bounded operator on* X *with* $r(\mathcal{C}_\nu) = \|\mathcal{C}_\nu\| = \frac{1}{|\Re(\nu-\gamma)|}$.

Proof: If $\Re(\nu) > \gamma$, then by Theorem 14, $\nu - \gamma \in \rho(\Gamma, X)$, and the resolvent operator is given by the Laplace transform (see Theorem 2 in Chapter 1, or [14, VIII.1.12]): For every $h \in X$,

$$R(\nu - \gamma, \Gamma)h = \int_0^\infty e^{-(\nu-\gamma)t} T_t h \, dt,$$

with convergence in norm. Applying change of variables: Let $\omega = e^{-t}z$, then $t = 0 \to \omega = z$, and $t = \infty \to \omega = 0$. Also, $t = \ln z - \ln \omega$ and $dt = -\frac{1}{\omega} d\omega$. Then

$$\begin{aligned} R(\nu - \gamma, \Gamma)h(z) &= -\int_0^z e^{-(\nu-\gamma)[\ln z - \ln \omega]} \left(\frac{\omega}{z}\right)^\gamma h(\omega) \left(\frac{-1}{\omega}\right) d\omega \\ &= z^{-(\nu-\gamma)-\gamma} \int_0^z \omega^{\nu-\gamma+\gamma-1} h(\omega) \, d\omega, \quad \text{as desired.} \end{aligned}$$

Alternatively, if $f = R(\nu - \gamma, \Gamma)h$, then f solves the differential equation

$$f'(z) + \frac{\nu}{z} f(z) = \frac{h(z)}{z} h \quad (z \in \mathbb{U}),$$

which implies that $(z^\nu f)'(z) = z^{(\nu-1)} h(z)$. Integrating along the path $\omega = e^{-t}z$, $0 \le t < \infty$, will yield the desired result.

Similarly, in the case that $\Re(\nu) < \gamma$, then $-(\nu - \gamma) \in \rho(-\Gamma, X)$, where $-\Gamma$ generates the strongly continuous semigroup $T_{-t}h(z) = e^{\gamma t} h(e^t z)$, $t \ge 0$ (see section 1.4.2 of Chapter 1). Again, the resolvent operator is given by the Laplace transform,

$$R(\nu - \gamma, \Gamma)h = -R(-(\nu - \gamma), -\Gamma)h = -\int_0^\infty e^{(\nu-\gamma)t} T_{-t} h \, dt,$$

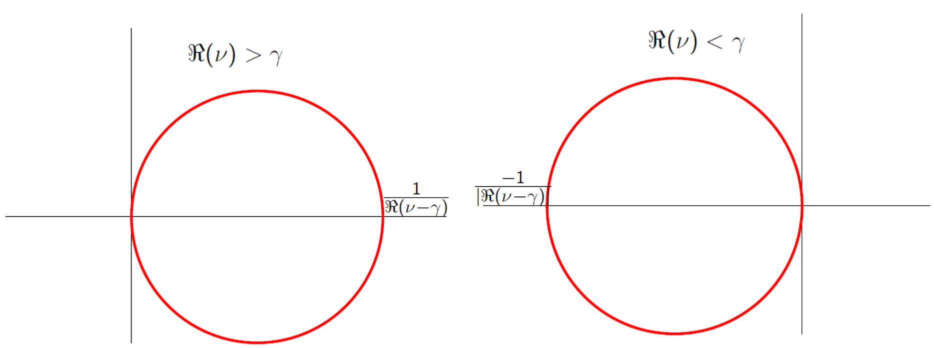
with convergence in norm. By change of variables, let $\omega = e^t z$, then

$$R(\nu - \gamma, \Gamma)h(z) = -\int_z^\infty e^{(\nu-\gamma)[\ln\omega - \ln z]} \left(\frac{\omega}{z}\right)^\gamma h(\omega)\frac{1}{\omega}\, d\omega$$
$$= -z^{-(\nu-\gamma)-\gamma} \int_z^\infty \omega^{\nu-\gamma+\gamma-1} h(\omega)\, d\omega, \qquad \text{as claimed.}$$

Now, since $\mathcal{C}_\nu = R(\nu - \gamma, \Gamma)$, then from the spectral mapping theorem (Theorem 1), and Proposition 14, it follows that

$$\sigma(\mathcal{C}_\nu, X) = \{\tfrac{1}{\nu-\gamma-it} : t \in \mathbb{R}\} \cup \{0\}$$
$$= \left\{\omega \in \mathbb{C} : |\omega - \tfrac{1}{2\Re(\nu-\gamma)}| = \tfrac{1}{2|\Re(\nu-\gamma)|}\right\},$$

which proves (a). See Figure 4.1 for a graphic representation of the $\sigma(\mathcal{C}_\nu)$. Since C_ν has spectral radius $r(\mathcal{C}_\nu) = \frac{1}{|\Re(\nu-\gamma)|}$, it follows from Hille-Yosida theorem (Theorem 2), that $\frac{1}{|\Re(\nu-\gamma)|} = r(\mathcal{C}_\nu) \leq \|\mathcal{C}_\nu\| \leq \frac{1}{|\Re(\nu-\gamma)|}$, and so (b) holds as well. ■



(a) (b)

Figure 4.1

Spectrum of the resolvent operator $\mathcal{C}_\nu = R(\nu - \gamma, \Gamma)$: (a) $\Re(\nu) > \gamma$, (b) $\Re(\nu) < \gamma$.

4.2 Adjoints of the Resolvent and Cesàro-like operators

We end this section with a representation of the adjoints of the resolvent operators $\mathcal{C}_\nu$ on the weighted Bergman and Hardy spaces. Let $1 < p < \infty$ and let q be conjugate to p in the sense that $\frac{1}{p} + \frac{1}{q} = 1$. We will write $\gamma_p = (\alpha+2)/p$, $\alpha \geq -1$ to make the dependence on p apparent. Following Chapter 2, we have that the dual space X^* of $X = H^p(\mathbb{U})$ or $L^p_a(\mathbb{U})$, $1 < p < \infty$ is given by $X^* \approx H^q(\mathbb{U})$ or $L^q_a(\mathbb{U}, \mu_\alpha)$ respectively under the duality pairings given by equations (2.9) and (2.13). It is important to note that under those pairings, the adjoint operator from $\mathcal{L}(X)$ to $\mathcal{L}(X^*)$ is conjugate linear. If $1 < r < \infty$, let $\gamma_r = \frac{\alpha+2}{r}$ and $\alpha \geq -1$. We first give the following result which gives the adjoint of the group $(T_t)_{t\in\mathbb{R}}$,

Theorem 13

Let X be one of the spaces $H^p(\mathbb{U})$ or $L^p_a(\mathbb{U}, \mu_\alpha)$, $1 < p < \infty$. Let $T_t f(z) = e^{-\gamma_p t} f(e^{-t}z)$ for every $f \in X$, and define $T_{-t}g(z) = e^{\gamma_q t} g(e^t z)$ for every $g \in X^*$. Then $(T_t)_{t\in\mathbb{R}}$ and $(T_{-t})_{t\in\mathbb{R}}$ are adjoints of each other, that is, $T_t^* = T_{-t}$.

Proof: First, we consider the case when $X = H^p(\mathbb{U})$, $1 < p < \infty$. Then $X^* = H^q(\mathbb{U})$ with $\frac{1}{p} + \frac{1}{q} = 1$, and

$$\begin{aligned}
\langle T_t f, g\rangle &= \int_{\mathbb{R}} e^{-\frac{t}{p}} f(e^{-t}x)\overline{g(x)}\, dx \;=\; \int_{\mathbb{R}} e^{-\frac{t}{p}} f(y)\overline{g(e^t y)} e^t\, dy \\
&= \int_{\mathbb{R}} f(y) e^{(1-\frac{1}{p})t}\overline{g(e^t y)}\, dy \;=\; \int_{\mathbb{R}} f(y)\overline{e^{\gamma_q t} g(e^t y)}\, dy \;=\; \langle f, T_{-t} g\rangle.
\end{aligned}$$

If $X = L^p_a(\mathbb{U}, \mu_\alpha)$, $1 < p < \infty$, $\alpha > -1$, then $X^* = L^q_a(\mathbb{U}, \mu_\alpha)$ and

$$\begin{aligned}\langle T_t f, g\rangle &= \int_{\mathbb{U}} e^{-\gamma_p t} f(e^{-t}z)\overline{g(z)}(\Im(z))^\alpha \, dA(z) \\ &= \int_{\mathbb{U}} e^{-\gamma_p t} f(\omega)\overline{g(e^t\omega)}(e^t\Im(\omega))^\alpha e^{2t} \, dA(\omega) \\ &= \int_{\mathbb{U}} e^{(\alpha+2)[1-\frac{1}{p}]t} f(\omega)\overline{g(e^t\omega)}(\Im(\omega))^\alpha \, dA(\omega) \\ &= \int_{\mathbb{U}} f(\omega)\overline{e^{\gamma_q t} g(e^t\omega)} \, d\mu_\alpha(\omega) \;=\; \langle f, T_{-t} g\rangle, \quad \text{as desired.}\end{aligned}$$

■

Since X is reflexive, it follows from Propositions 1 and 3 that $\Gamma_p^* = -\Gamma_q$ and that $R(\lambda, \Gamma_p)^* = R(\bar{\lambda}, -\Gamma_q)$. Then we obtain the adjoints of the resolvent operator $\mathcal{C}_\nu$ as given in the following theorem,

Theorem 14

Let X be one of the spaces $H^p(\mathbb{U})$ or $L^p_a(\mathbb{U}, \mu_\alpha)$, $1 < p < \infty$, $\alpha > -1$ ($\alpha = -1$ for $X = H^p(\mathbb{U})$), and let q be conjugate to p. If $\Re(\nu) \neq \gamma_p$, then

$$\mathcal{C}_\nu^* = -C_{\alpha+2-\bar{\nu}}.$$

Proof: Suppose $\Re(\nu) > \gamma_p$. Then $\nu-\gamma_p = \nu-\frac{\alpha+2}{p} = \nu-(\alpha+2)(1-\frac{1}{q}) = \nu-(\alpha+2)+\gamma_q > 0$, and

$$\begin{aligned}C_\nu^* &= R(\nu - \gamma_p, \Gamma_p)^* = R(\bar{\nu} - \gamma_p, \Gamma_p^*) \\ &= R(\bar{\nu} - (\alpha + 2) + \gamma_q, -\Gamma_q) \\ &= -R((\alpha + 2 - \bar{\nu}) - \gamma_q, \Gamma_q) = -C_{\alpha+2-\bar{\nu}}.\end{aligned}$$

Similarly, if $\Re(\nu) < \gamma_p$, then $\nu - \gamma_p = \nu - (\alpha + 2) + \gamma_q < 0$, and we therefore have

$$\begin{aligned} C_\nu^* &= -R(-(\nu - \gamma_p), -\Gamma_p)^* = -R(-(\bar{\nu} - \gamma_p), -\Gamma_p^*) \\ &= -R(-(\bar{\nu} - (\alpha + 2) + \gamma_q), \Gamma_q) \\ &= -R((\alpha + 2 - \bar{\nu}) - \gamma_q, \Gamma_q) = -C_{\alpha+2-\bar{\nu}}, \quad \text{as desired.} \end{aligned}$$

∎

CHAPTER 5

TRANSLATION AND ROTATION GROUPS

In this chapter, we present a detailed analysis of both the translation and rotation groups. We obtain the infinitesimal generators associated with each group, then study their spectral properties. Just as in the case of the scaling group, the resolvent operators are then obtained and represented concretely as integral operators. Using both spectral and semigroup theory of operators on Banach spaces, we study the norm and spectral properties of the obtained resolvent operators as well as their adjoints. We shall end each section by completing the analysis of the specific examples (Examples 2 and 3), that were earlier introduced in Chapter 3.

5.1 The Translation group

As noted before, for this group we consider $\varphi_t(z) = z + kt$, for all $t, k \in \mathbb{R},\ k \neq 0$ and $z \in \mathbb{U}$. Then for all $f \in X$, the corresponding group of invertible isometries is given by

$$T_t f(z) := S_{\varphi_t} f(z) = (\varphi_t'(z))^\gamma f(\varphi_t(z)) = f(z + kt),$$

where $\gamma = (\alpha + 2)/p$ as defined in the Chapter 4. We shall denote the generator of $(T_t)_{t\in\mathbb{R}}$ by Γ_k, while X is defined as before. Then we immediately obtain the generator Γ_k and its domain in the following theorem,

Theorem 15

Let X denote one of the spaces $H^p(\mathbb{U})$ or $L^p_a(\mathbb{U}, \mu_\alpha)$, $1 \leq p < \infty$. Then the infinitesimal generator Γ_k of the group $(T_t)_{t\in\mathbb{R}}$ is $\Gamma_k f(z) = kf'(z)$ with domain $\mathcal{D}(\Gamma_k) = \{f \in X : f' \in X\}$.

Proof: If $f \in \mathcal{D}(\Gamma_k)$, then, just as in the proof of Proposition 12,

$$\begin{aligned}\Gamma_k f(z) &= \lim_{t\to 0^+} \frac{f(z+kt) - f(z)}{t} = k\frac{\partial}{\partial t} f(z+kt)\Big|_{t=0} \\ &= kf'(z).\end{aligned}$$

Thus $\mathcal{D}(\Gamma_k) \subseteq \{f \in X : f' \in X\}$. Conversely, if $f \in X$ is such that $f' \in X$, then for all $z \in \mathbb{U}$, we have

$$\begin{aligned}T_t(f)(z) - f(z) &= \int_0^t \frac{\partial}{\partial s}\left(f(z+ks)\right) ds = \int_0^t kf'(\varphi_s(z))ds \\ &= \int_0^t T_s F(z) ds, \quad \text{where } F(z) = kf^{'}(z).\end{aligned}$$

Thus,

$$\lim_{t\to 0^+} \frac{T_t(f) - f}{t} = \lim_{t\to 0^+} \frac{1}{t}\int_0^t T_s F\, ds,$$

and strong continuity of $(T_s)_{s\geq 0}$ implies that $\frac{1}{t}\int_0^t \|T_s F - F\|\, ds \to 0$ as $t \to 0^+$. Thus $\{f \in X : f' \in X\} \subseteq \mathcal{D}(\Gamma_k)$. ∎

Proposition 15

Let X denote one of the spaces $H^p(\mathbb{U})$ or $L^p_a(\mathbb{U}, \mu_\alpha)$, $1 \leq p < \infty$. Then for all $f \in X$,

1. *$\Gamma_k f = kf' = k\Gamma_1 f$ with domain $\mathcal{D}(\Gamma_k) = \mathcal{D}(\Gamma_1) = \{f \in X : f' \in X\}$.*
2. *$\sigma(\Gamma_k) = k\sigma(\Gamma_1)$, and $\sigma_p(\Gamma_k) = k\sigma_p(\Gamma_1)$. In particular, $\lambda \in \rho(\Gamma_1)$ if and only if $k\lambda \in \rho(\Gamma_k)$, and*
$$R(k\lambda, \Gamma_k) = \frac{1}{k} R(\lambda, \Gamma_1) \qquad (5.1)$$

Proof: The first claim is clear from Theorem 15. For the second claim, suppose $\lambda \in \rho(\Gamma_1)$, then

$$(k\lambda - \Gamma_k)\,\frac{1}{k}R(\lambda,\Gamma_1) = (k\lambda - k\Gamma_1)\,\frac{1}{k}R(\lambda,\Gamma_1) = \frac{k}{k}\,(\lambda - \Gamma_1)\,R(\lambda,\Gamma_1) = I,$$

and if $f \in \mathcal{D}(\Gamma_1)$, then

$$\frac{1}{k}R(\lambda,\Gamma_1)\,(k\lambda - \Gamma_k)\,f = \frac{k}{k}R(\lambda,\Gamma_1)\,(\lambda - \Gamma_1)\,f = f.$$

Thus, $\lambda \in \rho(\Gamma_1) \;\Rightarrow\; k\lambda \in \rho(\Gamma_k)$ and equation (5.1) holds.

Conversely, if $k\lambda \in \rho(\Gamma_k)$, then

$$(\lambda - \Gamma_1)\,(kR(k\lambda,\Gamma_k)) = (k\lambda - \Gamma_k)\,R(k\lambda,\Gamma_k) = I,$$

and if $f \in \mathcal{D}(\Gamma_k)$, then

$$kR(k\lambda,\Gamma_k)\,(\lambda - \Gamma_1)\,f = R(k\lambda,\Gamma_k)\,(k\lambda - \Gamma_k)\,f = f,$$

which completes the proof. ∎

As a result of the above Proposition 15, we may restrict our attention to the case $\Gamma = \Gamma_1$. But we first prove the following lemma which gives the growth condition for f' whenever $f \in X$,

Lemma 4

For each of the spaces $X = H^p(\mathbb{U})$ or $X = L^p_a(\mathbb{U},\mu_\alpha)$, $1 \le p < \infty$, there is a constant K so that for every $f \in X$,

$$|f'(\omega)| \le \frac{K\|f\|}{\Im(\omega)^{\gamma+1}}. \tag{5.2}$$

Proof: The generalized Bloch space $\mathcal{B}_\infty^{1+\gamma}(\mathbb{D})$ is the space of functions $g \in \mathcal{H}(\mathbb{D})$ for which the seminorm $\|g\|_{\mathcal{B}_\infty^{1+\gamma}(\mathbb{D}),1} = \sup_{z\in\mathbb{D}}(1-|z|^2)^\gamma|g(z)| < \infty$. An equivalent seminorm is $\|g\|_{\mathcal{B}_\infty^{1+\gamma}(\mathbb{D}),2} = \sup_{z\in\mathbb{D}}(1-|z|^2)^{\gamma+1}|g'(z)|$, [40, Proposition 7]. Moreover, the spaces $X(\mathbb{D})$ are continuously embedded in $\mathcal{B}_\infty^{1+\gamma}(\mathbb{D})$. If $f \in X$, then $S_\psi f \in X(\mathbb{D})$ and

$$(S_\psi f)'(z) = \frac{2\gamma}{1-z}(\psi'(z))^\gamma f(\psi(z)) + (\psi'(z))^{\gamma+1} f'(\psi(z)).$$

Thus

$$\begin{aligned}
&\|S_\psi f\|_{\mathcal{B}_\infty^{1+\gamma}(\mathbb{D}),2} + 4\gamma\|S_\psi f\|_{\mathcal{B}_\infty^{1+\gamma}(\mathbb{D}),1} \\
&\qquad \geq \|S_\psi f\|_{\mathcal{B}_\infty^{1+\gamma}(\mathbb{D}),2} + 2\gamma \sup_{z\in\mathbb{D}}\left(\frac{1-|z|^2}{|1-z|}(1-|z|^2)^\gamma|S_\psi f(z)|\right) \\
&\qquad \geq \sup_{z\in\mathbb{D}}(1-|z|^2)^{\gamma+1}|f'(\psi(z))|.
\end{aligned}$$

Since $|\psi'(\psi^{-1}(\omega))|(1-|\psi^{-1}(\omega)|^2) = 2\Im(\omega)$, it follows that for some K depending on X,

$$K\|f\| \geq \sup_{\omega\in\mathbb{U}} \Im(\omega)^{\gamma+1}|f'(\omega)|.$$

■

A similar bound holds for the corresponding spaces on the disc. Specifically, similarly, for each $X(\mathbb{D})$ there is a constant K such that

$$|g'(z)| \leq \frac{K\|g\|}{(1-|z|^2)^{\gamma+1}} \tag{5.3}$$

for every $g \in X(\mathbb{D})$ and $z \in \mathbb{D}$.

From now henceforth we focus on $\Gamma f = \Gamma_1 f = f'$ for every $f \in X$. Then we obtain the next lemma.

Lemma 5

Let X denote one of the spaces $H^p(\mathbb{U})$ or $L^p_a(\mathbb{U}, \mu_\alpha)$, $1 \leq p < \infty$. If $\lambda \in \mathbb{C}$ is such that $\Im(\lambda) < 0$, then $\lambda \in \rho(\Gamma, X)$, and the resolvent operator $R(\lambda, \Gamma)$ on X is given by

$$R(\lambda, \Gamma)h(z) = e^{\lambda z} \int_z^\infty e^{-\lambda \omega} f(\omega)\, d\omega \quad (z \in \mathbb{U}).$$

Moreover,

$$\|R(\lambda, \Gamma)h(z)\| \leq \frac{1}{|\lambda|}. \tag{5.4}$$

Proof: Let $\lambda \in \mathbb{C}$ and $f, h \in \mathcal{H}(\mathbb{U})$. Then the equation $(\lambda - \Gamma)f = h$ is equivalent to the differential equation

$$(e^{-\lambda z} f(z))' = -e^{-\lambda z} h(z) \quad (z \in \mathbb{U}). \tag{5.5}$$

For $\Im(\lambda) < 0$, and irrespective of the value of $\Re(\lambda)$, we obtain the resolvent operator $f = R(\lambda, \Gamma)h$ by integrating equation (5.5) along the path $z + \frac{\bar{\lambda}}{|\lambda|}t$, $0 \leq t < \infty$.

Specifically, fix $\lambda = a + bi$ with $b = \Im(\lambda) < 0$ and let $h \in X$. Let $f \in \mathcal{H}(\mathbb{U})$ be given by

$$f(z) = e^{\lambda z} \int_z^\infty e^{-\lambda \omega} h(\omega)\, d\omega. \tag{5.6}$$

By integrating along the path $\omega = z + \frac{\bar{\lambda}}{|\lambda|}t$, $0 \leq t < \infty$, we have $\frac{\bar{\lambda}}{|\lambda|} = \frac{a-bi}{\sqrt{a^2+b^2}}$, $d\omega = \frac{\bar{\lambda}}{|\lambda|}\, dt$, and so

$$\begin{aligned} f(z) &= e^{\lambda z} \int_0^\infty e^{-\lambda(z + \frac{\bar{\lambda}}{|\lambda|}t)} h(\omega(t)) \frac{\bar{\lambda}}{|\lambda|}\, dt \\ &= \int_0^\infty e^{-|\lambda| t} h(\omega(t)) \frac{\bar{\lambda}}{|\lambda|}\, dt, \end{aligned}$$

where the integral is absolutely convergent by the growth condition (2.2). Fix $\omega \in \mathbb{U}$. Then for all $z \neq \omega$ with $\Im(z) \geq \frac{1}{2}\Im(\omega)$, Lemma 4 implies that

$$\left|\frac{f(z)-f(\omega)}{z-\omega}\right| \leq \left(\frac{2}{\Im(\omega)}\right)^{\gamma+1} K\|h\|,$$

and so a standard dominated convergence argument implies that $f \in \mathcal{H}(\mathbb{U})$. The function f also satisfies equation (5.5). Indeed, let

$$\begin{aligned} g(z) &= e^{\lambda z}\left(\int_z^i e^{-\lambda\omega} h(\omega)\, d\omega + \int_i^\infty e^{-\lambda\omega} h(\omega)\, d\omega\right) \\ &= e^{\lambda z}\int_0^\infty e^{-\lambda\omega(t)} h(\omega(t))\, d\omega(t), \end{aligned}$$

where

$$\omega(t) = \begin{cases} (1-t)z + it & \text{if } 0 \leq t \leq 1 \\ i + \frac{\bar{\lambda}}{|\lambda|}(t-1) & \text{if } t > 1. \end{cases}$$

Then g satisfies equation (5.5), and by Cauchy's theorem,

$$\begin{aligned} &\left|e^{\lambda z}\int_0^u e^{-\lambda\omega(t)} h(\omega(t))\, d\omega(t) - \frac{\bar{\lambda}}{|\lambda|}\int_0^u e^{-|\lambda| t} h(z + \frac{\bar{\lambda}}{|\lambda|}t)\, dt\right| \\ &\qquad = \left|e^{\lambda z}\int_{i+\frac{\bar{\lambda}}{|\lambda|}u}^{z+\frac{\bar{\lambda}}{|\lambda|}u} e^{-\lambda\xi} h(\xi)\, d\xi\right| \to 0 \end{aligned}$$

as $u \to \infty$ by the growth condition (2.2). Thus if $\Im(\lambda) < 0$, then

$$f(z) = R(\lambda, \Gamma)h(z) = e^{\lambda z}\int_z^\infty e^{-\lambda\omega} h(\omega)\, d\omega, \qquad \text{as claimed.}$$

Finally, we prove the last assertion of the lemma. For every $t > 0$, Proposition 5 implies that the composition operator $h(z) \mapsto h(z + \frac{\bar{\lambda}}{|\lambda|}t)$ is a contraction on X.

In the case that $X = L^p_a(\mathbb{U}, \mu_\alpha)$, then, by the integral version of Minkowski's inequality, we have

$$\begin{aligned}
\|f\|_{L^p(\mu_\alpha)} &= \left(\int_{\mathbb{U}} \left(\left|\frac{\bar{\lambda}}{|\lambda|}\right| \int_0^\infty e^{-|\lambda|t} \left|h(z + \frac{\bar{\lambda}}{|\lambda|}t)\right| \, dt\right)^p d\mu_\alpha(z)\right)^{\frac{1}{p}} \\
&= \int_0^\infty e^{-|\lambda|t} \left(\int_{\mathbb{U}} \left|h(z + \frac{\bar{\lambda}}{|\lambda|}t)\right|^p d\mu_\alpha(z)\right)^{\frac{1}{p}} dt,
\end{aligned}$$

and

$$\|h(z + \frac{\bar{\lambda}}{|\lambda|}t)\|_{L^p(\mu_\alpha)} \leq \|h\|_{L^p(\mu_\alpha)}.$$

Thus

$$\|f\|_{L^p(\mu_\alpha)} \leq \int_0^\infty e^{-|\lambda|t} \|h\| \, dt = \frac{1}{|\lambda|} \|h\|_{L^p(\mu_\alpha)}.$$

A similar computation shows that if $X = H^p(\mathbb{U})$, and $h \in H^p(\mathbb{U})$, then $f \in H^p(\mathbb{U})$ with

$$\|f\|_{H^p(\mathbb{U})} \leq \frac{1}{|\lambda|} \|h\|_{H^p(\mathbb{U})}.$$

Since $f = R(\lambda, \Gamma)h$, it immediately follows that whenever $\Im(\lambda) < 0$, $R(\lambda, \Gamma) \in \mathcal{L}(X)$ and $\|R(\lambda, \Gamma)\| \leq \frac{1}{|\lambda|}$ which completes our proof. ■

We now establish some spectral properties of the generator Γ in the following result;

Theorem 16

Let X denote one of the spaces $H^p(\mathbb{U})$ or $L^p_a(\mathbb{U}, \mu_\alpha)$, $1 \leq p < \infty$. Then $\sigma_p(\Gamma, X) = \emptyset$ and $\sigma(\Gamma, X) = \{is : s \geq 0\}$. In particular Γ is an unbounded operator.

Proof: Let $\lambda \in \mathbb{C}$ be such that $\lambda \in \sigma_p(\Gamma, X)$. Then $\Gamma f(z) = \lambda f(z)$ for some $0 \neq f \in X$, which is equivalent to the differential equation $f'(z) - \lambda f(z) = 0$, whose general solution is $f(z) = ce^{\lambda z}$ for some constant c. But by Lemma 3, $ce^{\lambda z} \in X$ only if $c = 0$. This implies that $f = 0$ which is a contradiction and thus $\sigma_p(\Gamma, X) = \emptyset$.

Since each T_t is an invertible isometry, just as in Proposition 14, its spectrum satisfies $\sigma(T_t) \subseteq \partial\mathbb{D}$, and together with the spectral mapping theorem for strongly continuous groups, we easily conclude that $\sigma(\Gamma, X) \subseteq i\mathbb{R}$.

To establish that $\{is : s \geq 0\} \subseteq \sigma(\Gamma, X)$, let $\lambda \in \mathbb{C}$ and $f, h \in \mathcal{H}(\mathbb{U})$. Then the equation $(\lambda - \Gamma)f = h$ is equivalent to the differential equation (5.5). Now, let $\lambda = is$ for some $s \in \mathbb{R},\ s \geq 0$. We consider the Bergman and Hardy spaces separately.

For $X = L^p_a(\mathbb{U}, \mu_\alpha)$, let c satisfy $1/p < c < \min\{1 + 1/p, \gamma\}$, $c \neq 1$, so that $h(z) = e^{\lambda z}/z^c \in L^p_a(\mathbb{U}, \mu_\alpha)$ by Lemma 3. Then by equation (5.5), we obtain

$$\begin{aligned} f(z) &= -e^{\lambda z} \int e^{-\lambda z} \cdot e^{\lambda z}/z^c \, dz = -e^{\lambda z} \int z^{-c} \, dz \\ &= K e^{\lambda z} - (1 - c)^{-1} e^{\lambda z}/z^{c-1}, \end{aligned}$$

for some constant K. But by Lemma 3, $K e^{\lambda z} - (1 - c)^{-1} e^{\lambda z}/z^{c-1} \notin L^p_a(\mathbb{U}, \mu_\alpha)$ for any K, and therefore $h \notin \mathcal{R}(\lambda - \Gamma)$. Thus $\{is : s \geq 0\} \subseteq \sigma(\Gamma, L^p_a(\mathbb{U}, \mu_\alpha))$.

In the Hardy space case, let $h(z) = e^{\lambda z}(z + i)^{-(1+1/p)}$. Then $h \in H^p(\mathbb{U})$ by Lemma 3, and equation (5.5) yields

$$\begin{aligned} f(z) &= -e^{\lambda z} \int e^{-\lambda z} \cdot e^{\lambda z}(z + i)^{-(1+1/p)} \, dz \\ &= K e^{\lambda z} + p e^{\lambda z}(z + i)^{-\frac{1}{p}}, \end{aligned}$$

for some constant K. Again, by the Lemma 3, $K e^{\lambda z} + p e^{\lambda z}(z + i)^{-\frac{1}{p}} \notin H^p(\mathbb{U})$ for any K, and therefore $(\lambda - \Gamma)$ is not surjective. Thus $\{is : s \geq 0\} \subseteq \sigma(\Gamma, H^p(\mathbb{U}))$, as desired. Therefore following Lemma 5, we conclude that $\sigma(\Gamma, X) = \{is : s \geq 0\}$. ■

Because we know the spectrum of the generator $\sigma(\Gamma)$, we can now obtain the representation of the resolvents.

Theorem 17

Let X denote one of the spaces $H^p(\mathbb{U})$ or $L^p_a(\mathbb{U},\mu_\alpha)$, $1 \le p < \infty$. If $\lambda \in \rho(\Gamma, X) := \mathbb{C}\backslash\sigma(\Gamma, X)$, then the resolvent operator $R(\lambda,\Gamma)$ on X is given concretely by

$$R(\lambda,\Gamma)h(z) = e^{\lambda z}\int_z^\infty e^{-\lambda\omega}h(\omega)\,d\omega \quad (z \in \mathbb{U}).$$

Proof: Recall from Theorem 16 that the spectrum of Γ is given by, $\sigma(\Gamma, X) = \{is : s \ge 0\}$. For $\lambda \notin \sigma(\Gamma, X)$, we consider separately the following two cases: $\Im(\lambda) \ge 0$ and $\Im(\lambda) < 0$. For the case $\Im(\lambda) < 0$, we refer to Lemma 5.

Now, for $\Im(\lambda) \ge 0$, we have $\Re(\lambda) \ne 0$, otherwise $\lambda \notin \rho(\Gamma, X)$. Therefore,

(i) If $\Re(\lambda) > 0$, then

$$R(\lambda,\Gamma)h(z) = \int_0^\infty e^{-\lambda t}T_t h(z)\,dt = \int_0^\infty e^{-\lambda t}h(z+t)\,dt.$$

By change of variables; letting $\omega = z + t$, then

$$\begin{aligned} R(\lambda,\Gamma)h(z) &= \int_z^\infty e^{-\lambda(\omega - z)}h(\omega)\,d\omega \\ &= e^{\lambda z}\int_z^\infty e^{-\lambda\omega}h(\omega)\,d\omega. \end{aligned}$$

(ii) If $\Re(\lambda) < 0$, then $R(\lambda,\Gamma)h = -R(-\lambda,-\Gamma)h$ and thus,

$$\begin{aligned} R(\lambda,\Gamma)h(z) = -\int_0^\infty e^{-(-\lambda t)}T_{-t}h(z)\,dt &= -\int_0^\infty e^{\lambda t}T_{-t}h(z)\,dt \\ &= -\int_0^\infty e^{\lambda t}h(z-t)\,dt. \quad (5.7) \end{aligned}$$

Changing variables by letting $\omega = z - t$ yields

$$\begin{aligned} R(\lambda,\Gamma)h(z) &= -\int_z^\infty e^{\lambda(z-\omega)}h(\omega)\,(-d\omega) \\ &= e^{\lambda z}\int_z^\infty e^{-\lambda\omega}h(\omega)\,d\omega. \end{aligned}$$

■

The spectrum of the resolvent operators $R(\lambda, \Gamma)$ on X turns out to be given in terms of circular arcs as we give in the next proposition.

Proposition 16

If $\Re(\lambda) \neq 0$, denote the circle $\left|z - \frac{1}{2\Re(\lambda)}\right| = \frac{1}{|2\Re(\lambda)|}$ by C_λ, and if $\lambda = ib$ for some $b < 0$, take C_λ to be the imaginary axis.

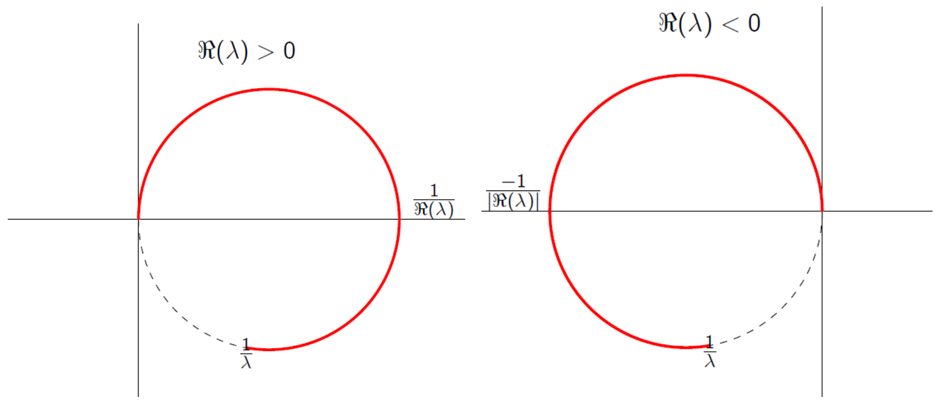
1. *If $\Re(\lambda) \neq 0$ and $\Im(\lambda) \geq 0$, then $\sigma(R(\lambda, \Gamma))$ is the arc of the circle C_λ from $\frac{1}{\lambda}$ to 0 that contains the upper half of C_λ. Moreover, $\|R(\lambda, \Gamma)\| = r(R(\lambda, \Gamma)) = \frac{1}{|\Re(\lambda)|}$.*
2. *If $\Im(\lambda) < 0$, then $\sigma(R(\lambda, \Gamma))$ is the arc of the circle C_λ from $1/\lambda$ to 0 contained in the upper half of C_λ. In this case, $\|R(\lambda, \Gamma)\| = r(R(\lambda, \Gamma)) = \frac{1}{|\lambda|}$.*

Proof: The spectral mapping theorem and Theorem 15 imply that for all $\lambda \in \rho(\Gamma)$, $\sigma(R(\lambda, \Gamma)) = \left\{\frac{1}{\lambda - is} : s \geq 0\right\} \cup \{0\}$. Thus, if $\lambda \in \rho(\Gamma)$ with $\Im(\lambda) \geq 0$, then $\sigma(R(\lambda, \Gamma))$ is the arc of the circle C_λ from $\frac{1}{\lambda}$ to 0 that contains the upper half of C_λ, (see Figure 5.1 where the thick lines represent the spectrum), and $r(R(\lambda, \Gamma)) = \frac{1}{|\Re(\lambda)|}$. The Hille-Yosida theorem yields $r(R(\lambda, \Gamma)) \leq \|R(\lambda, \Gamma)\| \leq \frac{1}{|\Re(\lambda)|}$.

In case $\lambda \in \rho(\Gamma)$ has $\Im(\lambda) < 0$, then $\sigma(R(\lambda, \Gamma))$ is the arc of the circle C_λ from $1/\lambda$ to 0 contained in the upper half of C_λ, (see Figures 5.2 and 5.3), and $r(R(\lambda, \Gamma)) = \frac{1}{|\lambda|}$. The norm estimate for $R(\lambda, \Gamma)$ in Lemma 5 along with its spectral radius yields $r(R(\lambda, \Gamma)) \leq \|R(\lambda, \Gamma)\| \leq \frac{1}{|\lambda|}$. Moreover, if $\Re(\lambda) = 0$, and $\Im(\lambda) < 0$, we have $\frac{1}{|\Im(\lambda)|} = r(R(\lambda, \Gamma)) \leq \|R(\lambda, \Gamma)\| \leq \frac{1}{|\Im(\lambda)|}$.

■

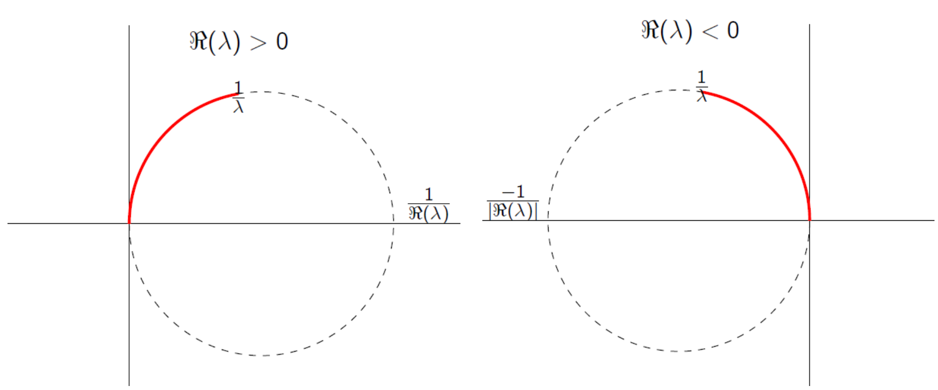
Let us denote the resolvent operator $R(\lambda, \Gamma)$ in this setting by $R_\lambda := R(\lambda, \Gamma)$. Clearly, from the above results R_λ is bounded on $X = H^p(\mathbb{U})$ or $L^p_a(\mathbb{U}, \mu_\alpha)$. We wish to study the



(a) (b)

Figure 5.1

Spectrum of the resolvent operator $R_\lambda = R(\lambda, \Gamma)$ when $\Im(\lambda) \geq 0$: (a) $\Re(\lambda) > 0$, (b) $\Re(\lambda) < 0$.



(a) (b)

Figure 5.2

Spectrum of the resolvent operator $R_\lambda = R(\lambda, \Gamma)$ when $\Im(\lambda) < 0$: (a) $\Re(\lambda) > 0$, (b) $\Re(\lambda) < 0$.

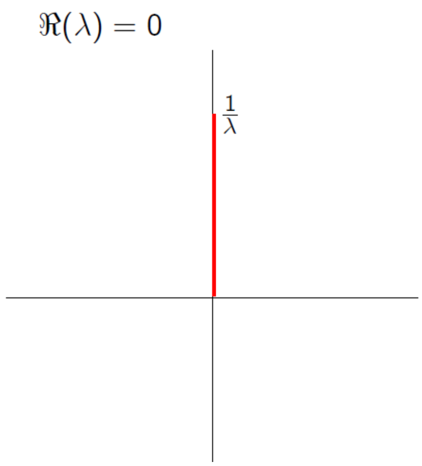


Figure 5.3

Spectrum of the resolvent operator $R_\lambda = R(\lambda, \Gamma)$ when $\Im(\lambda) < 0$ and $\Re(\lambda) = 0$.

adjoint operator R_λ^* of this resolvent operator. Let $1 < p, q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$ and write $\gamma_r = \frac{\alpha+2}{r}$, $\alpha \geq -1$ to make the dependence on p apparent.

Recall from Chapter 2 that $(H^p(\mathbb{U}))^* \approx H^q(\mathbb{U})$ and $(L_a^p(\mathbb{U}, \mu_\alpha))^* \approx L_a^q(\mathbb{U}, \mu_\alpha)$ under the sesquilinear pairings given by equations (2.13) and (2.9) respectively. Just as is the case in Chapter 3, we take note that under those pairings, the adjoint operator from $\mathcal{L}(X)$ to $\mathcal{L}(X^*)$ is conjugate linear. We first obtain the adjoint T_t^* of the group T_t.

Theorem 18

Let $X = H^p(\mathbb{U})$ or $L_a^p(\mathbb{U}, \mu_\alpha)$, $1 < p < \infty$. Let $T_t f(z) = f(z+t)$ for every $f \in X$, and define $T_{-t} g(z) = g(z-t)$ for all $g \in X^*$. Then $(T_t)_{t\in\mathbb{R}}$ and $(T_{-t})_{t\in\mathbb{R}}$ are adjoints of each other, that is, $T_t^* = T_{-t}$.

Proof: If $X = H^p(\mathbb{U})$, then $X^* = H^q(\mathbb{U})$ and for all $f \in X,\ g \in X^*$, $z = x + yi \in \mathbb{U}$, we have

$$\langle T_t f, g\rangle = \int_{\mathbb{R}} f(x+t)\overline{g(x)}\,dx = \int_{\mathbb{R}} f(u)\overline{g(u-t)}\,du = \langle f, T_t g\rangle.$$

For $X = L^p_a(\mathbb{U}, \mu_\alpha)$, we have $X^* = L^q_a(\mathbb{U}, \mu_\alpha)$ and for every $f \in X,\ z \in \mathbb{U}$,

$$\begin{aligned}\langle T_t f, g\rangle &= \int_{\mathbb{U}} f(z+t)\overline{g(z)}(\Im(z))^\alpha\,dA(z)\\ &= \int_{\mathbb{U}} f(\omega)\overline{g(\omega - t)}(\Im(\omega))^\alpha\,dA(\omega) = \langle f, T_{-t} g\rangle, \quad \text{as desired.}\end{aligned}$$

∎

Again, since X is reflexive, it follows from Propositions 1 and 3 that $R(\lambda, \Gamma)^* = R(\bar{\lambda}, \Gamma^*)$, and $\Gamma_p^* = -\Gamma_q$. We can now easily obtain the adjoint R_λ^* of the resolvent operator.

Theorem 19

Let X be one of the spaces $H^p(\mathbb{U})$ or $L^p_a(\mathbb{U}, \mu_\alpha)$, $1 < p < \infty$, $\alpha \geq -1$ ($\alpha = -1$ for $X = H^p(\mathbb{U})$), and let q be conjugate to p. If $\lambda \in \rho(\Gamma, X)$, then

$$R_\lambda^* = -R_{-\bar{\lambda}}.$$

Proof: If $\lambda \in \rho(\Gamma, X)$, then

$$\begin{aligned}R_\lambda^* &= R(\lambda, \Gamma_p)^* = R(\bar{\lambda}, \Gamma_p^*) = R(\bar{\lambda}, -\Gamma_q)\\ &= -R(-\bar{\lambda}, \Gamma_q) = -R_{-\bar{\lambda}}, \quad \text{as claimed.}\end{aligned}$$

∎

We end this section by revisiting Example 2 in Chapter 3, which as demonstrated corresponds to the translation group under consideration.

By the assumptions of the Example 2, we take $k = 1$, $g(z) = \frac{z}{1-z}$, and $g^{-1}(z) = \frac{z}{1+z}$, $\forall z \in \mathbb{U}$ in the translation group. Therefore, let $u_t(z) = z + t$. Then as in the Example 2,

$$\varphi_t(z) := \frac{(1-t)z + t}{-tz + 1 + t} = g^{-1}(g(z) + t)$$
$$= g^{-1} \circ u_t \circ g(z),$$

and so $S_{\varphi_t} = S_g S_{u_t} S_g^{-1}$. Following Theorems 15, 16, and 17, if Γ is the infinitesimal generator of the group S_{u_t} on X, then,

1. $\Gamma f(z) = f'(z)$ with domain $\mathcal{D}(\Gamma) = \{f \in X : f' \in X\}$,
2. $\sigma_p(\Gamma, X) = \emptyset$ and $\sigma(\Gamma, X) = \{is : s \geq 0\}$.
3. If $\lambda \in \rho(\Gamma)$, then $R(\lambda, \Gamma)h(z) = e^{\lambda z} \int_z^\infty e^{-\lambda\omega} h(\omega)\, d\omega := R_\lambda h(z)$.

We give the following proposition that details the analysis of the group in the example 2.

Proposition 17

Let $X = H^p(\mathbb{U})$ or $L_a^p(\mathbb{U}, \mu_\alpha)$, $1 \leq p < \infty$. Let $\varphi_t \in \mathrm{Aut}(\mathbb{U})$ be given by $\varphi_t(z) = \frac{(1-t)z+t}{-tz+1+t}$ for $t \in \mathbb{R}$, $z \in \mathbb{U}$, and S_{φ_t} be the corresponding group of isometries. Then,

1. *The infinitesimal generator Δ of $S_{\varphi_t} \subset \mathcal{L}(X)$ is given by*
$$\Delta(h(z)) = -2\gamma(1-z)h(z) + (1-z)^2 h'(z)$$
with domain $\mathcal{D}(\Delta) = \{h \in X : -2\gamma(1-z)h(z) + (1-z)^2 h'(z) \in X\}$.
2. *$\sigma_p(\Delta, X) = \emptyset$ and $\sigma(\Delta, X) = \{is : s \geq 0\}$.*
3. *If $\lambda \in \rho(\Delta)$, then*
$$R(\lambda, \Delta)h(z) = \frac{1}{(1-z)^{2\gamma}} e^{\lambda \frac{z}{1-z}} \int_z^\infty e^{\lambda \frac{\omega}{1-\omega}} (1-\omega)^{2\gamma-2} h(\omega)\, d\omega.$$

Proof: Since Γ is the generator of the group S_{u_t}, and $S_{\varphi_t} = S_g S_{u_t} S_g^{-1}$, it follows from Chapter 1, subsection 1.4.3, that the generator Δ of the group S_{φ_t} is given by

$$\Delta = S_g \Gamma S_g^{-1} \text{ with domain } \mathcal{D}(\Delta) = S_g \mathcal{D}(\Gamma).$$

Now, let $f' \in X$, then $f \in \mathcal{D}(\Gamma)$ and $h := S_g f$ belongs to $\mathcal{D}(\Delta)$ with $f = S_g^{-1} h$. Then

$$\begin{aligned} \Delta(h(z)) &= S_g \Gamma S_g^{-1} h(z) = S_g \Gamma f(z) = S_g f'(z) \\ &= (g'(z))^\gamma f'(g(z)) = \frac{1}{(1-z)^{2\gamma}} f'(g(z)). \end{aligned} \tag{5.8}$$

But $f(z) = S_g^{-1} h(z) = S_{g^{-1}} h(z) = \frac{1}{(1+z)^{2\gamma}} h\left(g^{-1}(z)\right)$, implying that

$$\begin{aligned} f'(z) &= -2\gamma(1+z)^{-2\gamma-1} h(g^{-1}(z)) + \frac{1}{(1+z)^{2\gamma+2}} h'(g^{-1}(z)) \\ &= (1+z)^{-2\gamma-2}\left(-2\gamma(1+z)h(g^{-1}(z)) + h'(g^{-1}(z))\right), \quad \text{so that} \end{aligned}$$

$$\begin{aligned} f'(g(z)) &= \left(1 + \frac{z}{1-z}\right)^{-2\gamma-2}\left(-2\gamma\left(1+\frac{z}{1-z}\right)h(z) + h'(z)\right) \\ &= (1-z)^{2\gamma}\left(-2\gamma(1-z)h(z) + (1-z)^2 h'(z)\right). \end{aligned}$$

Therefore, equation (5.8) becomes $\Delta(h(z)) = -2\gamma(1-z)h(z) + (1-z)^2 h'(z)$, as desired, with the domain

$$\begin{aligned} \mathcal{D}(\Delta) &= S_g \mathcal{D}(\Gamma) = \{S_g f : f \in \mathcal{D}(\Gamma)\} = \{h = S_g f : S_g f' \in X\} \\ &= \left\{h \in X : -2\gamma(1-z)h(z) + (1-z)^2 h'(z) \in X\right\}. \end{aligned}$$

Since $\Delta = S_g \Gamma S_g^{-1}$ and S_g is invertible, it's clear again from subsection 1.4.3 that $\sigma_p(\Delta, X) = \sigma_p(\Gamma, X) = \emptyset$ and $\sigma(\Delta, X) = \sigma(\Gamma, X) = \{is : s \geq 0\}$.

For the resolvents, we have: If $\lambda \in \rho(\Delta, X) = \rho(\Gamma, X)$, then

$$R(\lambda, \Delta) = S_g R(\lambda, \Gamma) S_g^{-1},$$

and thus we have,

$$\begin{aligned}
R(\lambda, \Gamma)h(z) = S_g R(\lambda, \Gamma) S_g^{-1} h(z) &= S_g \left(e^{\lambda z} \int_z^{\infty} e^{-\lambda \omega} S_g^{-1} h(\omega)\, d\omega \right) \\
&= S_g \left(e^{\lambda z} \int_z^{\infty} e^{-\lambda \omega} \frac{1}{(1+\omega)^{2\gamma}} h(g^{-1}(\omega))\, d\omega \right) \\
&= \frac{1}{(1-z)^{2\gamma}} e^{\lambda g(z)} \int_z^{\infty} e^{-\lambda g(\omega)} \frac{1}{(1+g(\omega))^{2\gamma}} h(\omega)\, dg(\omega) \\
&= \frac{1}{(1-z)^{2\gamma}} e^{\lambda \frac{z}{1-z}} \int_z^{\infty} e^{-\lambda \frac{\omega}{1-\omega}} (1-\omega)^{2\gamma-2} h(\omega)\, d\omega,
\end{aligned}$$

which completes the proof. ∎

5.2 The Rotation group

Again, as noted in Chapter 3, we consider the group $\varphi_t(z) = e^{ikt}z$, but modify the corresponding isometries by introducing a second parameter $c \in \mathbb{R}$. We therefore consider isometries on the spaces $X(\mathbb{D})$ of the form $T_t f(z) = e^{ict} f(e^{ikt}z)$ with $c,\ k \in \mathbb{R}$, $k \neq 0$. Then arguing as in Theorem 11, we see that $(T_t)_{t\in\mathbb{R}}$ is a strongly continuous group of isometries on $X(\mathbb{D})$. We denote its generator by $\Gamma_{c,k}$.

Recall that the operator $M_z f(z) := z f(z)$ is bounded and bounded below on each of the spaces $X(\mathbb{D}) = H^p(\mathbb{D})$, $1 \leq p < \infty$ or $X(\mathbb{D}) = L^p_a(\mathbb{D}, m_\alpha)$, $1 \leq p < \infty$ and $\alpha > -1$, with $\mathcal{R}(M_z) = \{f \in X(\mathbb{D}) : f(0) = 0\}$. The left inverse of M_z is the operator $Qf(z) := \frac{f(z)-f(0)}{z}$. For every $m \in \mathbb{N}$, $X(\mathbb{D}) = \mathcal{R}(M_z^m) \oplus \operatorname{span}\{z^n : n \in \mathbb{Z}_+,\ n < m\}$, and $P_m = M_z^m Q^m$ is the projection of $X(\mathbb{D})$ onto $\mathcal{R}(M_z^m)$ with kernel $\operatorname{span}\{z^n : n \in \mathbb{Z}_+,\ n < m\}$.

Theorem 20

Let $X(\mathbb{D})$ denote one of the spaces $H^p(\mathbb{D})$ or $L^p_a(\mathbb{D}, m_\alpha)$, $1 \leq p < \infty$. Then the infinitesimal generator $\Gamma_{c,k}$ of the group $(T_t)_{t\in\mathbb{R}} \subset \mathcal{L}(X(\mathbb{D}))$ is $\Gamma_{c,k} f(z) = i\,(cf(z) + kzf'(z))$ with domain $\mathcal{D}(\Gamma_{c,k}) = \{f \in X(\mathbb{D}) : f' \in X(\mathbb{D})\}$.

Proof: As in the previous cases,

$$\begin{aligned}\Gamma_{c,k} f(z) &= \frac{\partial}{\partial_t}\left(e^{ict} f(e^{ikt}z)\right)\Big|_{t=0}\\ &= \left(ice^{ict} f(e^{ikt}z) + e^{ict} ike^{ikt} zf'(e^{ikt}z)\right)\Big|_{t=0}\\ &= i(cf(z) + kzf'(z)).\end{aligned}$$

Therefore the domain $\mathcal{D}(\Gamma_{c,k}) \subset \{f \in X(\mathbb{D}) : zf' \in X(\mathbb{D})\}$. But $zf' \in X(\mathbb{D})$ implies that $zf' \in \mathcal{R}(M_z)$ and therefore $f' \in X(\mathbb{D})$. Thus $\{f \in X(\mathbb{D}) : zf' \in X(\mathbb{D})\} = \{f \in X(\mathbb{D}) : f' \in X(\mathbb{D})\}$.

Conversely if $f \in X(\mathbb{D})$ is such that $zf' \in X$, then $F(z) = i(cf(z) + kzf'(z)) \in X(\mathbb{D})$, and for all $t > 0$

$$\begin{aligned}\frac{T_t f(z) - f(z)}{t} &= \frac{1}{t}\int_0^t \partial_s(T_s f(z))\,ds\\ &= \frac{1}{t}\int_0^t e^{ics}\left[i(cf(e^{iks}z) + k(e^{iks}z)f'(e^{iks}z))\right]\,ds = \frac{1}{t}\int_0^t T_s F(z)\,ds.\end{aligned}$$

Again, strong continuity of $(T_s)_{s\geq 0}$ implies that

$$\left\|\frac{1}{t}\int_0^t T_s F\,ds - F\right\| \leq \frac{1}{t}\int_0^t \|T_s F - F\|\,ds \to 0 \text{ as } t \to 0^+.$$

Thus, $\mathcal{D}(\Gamma_{c,k}) = \{f \in X(\mathbb{D}) : f' \in X(\mathbb{D})\}$. ∎

Proposition 18

Let $X(\mathbb{D})$ denote one of the spaces $H^p(\mathbb{D})$ or $L^p_a(\mathbb{D}, m_\alpha)$, $1 \leq p < \infty$. Then

1. $\Gamma_{c,k} = ic + k\Gamma_{0,1}$ *with domain* $\mathcal{D}(\Gamma_{c,k}) = \mathcal{D}(\Gamma_{0,1}) = \{f : f' \in X(\mathbb{D})\}$ *.*

2. $\sigma(\Gamma_{c,k}) = \{ic + k\sigma(\Gamma_{0,1})\}$*, and* $\sigma_p(\Gamma_{c,k}) = \{ic + k\sigma_p(\Gamma_{0,1})\}$*.*

In fact, $\lambda \in \rho(\Gamma_{0,1})$ *if and only if* $ic + k\lambda \in \rho(\Gamma_{c,k})$*, and*

$$R(ic + k\lambda, \Gamma_{c,k}) = \frac{1}{k} R(\lambda, \Gamma_{0,1}). \tag{5.9}$$

Proof: From Theorem 20, $\Gamma_{0,1} f(z) = izf'(z)$ for all $f \in X(\mathbb{D})$. Therefore,

$$\Gamma_{c,k} f(z) = i\left(cf(z) + kzf'(z)\right) = icf(z) + k\Gamma_{0,1} f(z),$$

with same domain as claimed.

Now, let $\lambda \in \rho(\Gamma_{0,1})$, then

$$\begin{aligned}(ic + k\lambda - \Gamma_{c,k})\frac{1}{k}R(\lambda, \Gamma_{0,1}) &= \left(ic + k\lambda - (ic + k\Gamma_{0,1})\right)\frac{1}{k}R(\lambda, \Gamma_{0,1}) \\ &= \frac{k}{k}(\lambda - \Gamma_{0,1})R(\lambda, \Gamma_{0,1}) = I,\end{aligned}$$

and if $f \in \mathcal{D}(\Gamma_{c,k})$, then

$$\begin{aligned}\frac{1}{k}R(\lambda, \Gamma_{0,1})(ic + k\lambda - \Gamma_{c,k})f &= \frac{1}{k}R(\lambda, \Gamma_{0,1})\left(ic + k\lambda - (ic + k\Gamma_{0,1})\right)f \\ &= \frac{k}{k}R(\lambda, \Gamma_{0,1})(\lambda - \Gamma_{0,1})f = f.\end{aligned}$$

Conversely, if $\mu \in \rho(\Gamma_{c,k})$, let $\mu = ic + k\lambda$ so that $\lambda = \frac{\mu - ic}{k}$. Then

$$\begin{aligned}(\lambda - \Gamma_{0,1})kR(\mu, \Gamma_{c,k}) &= k\left(\frac{\mu - ic}{k} - \Gamma_{0,1}\right)R(\mu, \Gamma_{c,k}) = (\mu - ic - k\Gamma_{0,1})R(\mu, \Gamma_{c,k}) \\ &= \left(\mu - (ic + k\Gamma_{0,1})\right)R(\mu, \Gamma_{c,k}) = (\lambda - \Gamma_{c,k})R(\mu, \Gamma_{c,k}) = I,\end{aligned}$$

and if $f \in \mathcal{D}(\Gamma_{0,1})$, then

$$\begin{aligned}kR(\mu, \Gamma_{c,k})(\lambda - \Gamma_{0,1})f &= R(\mu, \Gamma_{c,k})(\mu - ic - k\Gamma_{0,1})f \\ &= R(\mu, \Gamma_{c,k})(\mu - \Gamma_{c,k})f = f.\end{aligned}$$

Thus, $\sigma(\Gamma_{c,k}) = \{ic + k\lambda : \lambda \in \sigma(\Gamma_{0,1})\}$, $\sigma_p(\Gamma_{c,k}) = \{ic + k\lambda : \lambda \in \sigma_p(\Gamma_{0,1})\}$, and for all $\lambda \in \rho(\Gamma_{0,1})$, $R(ic + k\lambda, \Gamma_{c,k}) = \frac{1}{k} R(\lambda, \Gamma_{0,1})$, as desired. ■

As a result of Proposition 18, from now on, we shall restrict our attention to the generator $\Gamma := \Gamma_{0,1}$, that is, for all $f \in X(\mathbb{D})$, $\Gamma f(z) = izf'(z)$. We therefore have the following results;

Theorem 21

Let $X(\mathbb{D})$ denote one of the spaces $H^p(\mathbb{D})$ or $L^p_a(\mathbb{D}, m_\alpha)$, $1 \leq p < \infty$ and $\alpha > -1$. Then

1. $\sigma(\Gamma, X(\mathbb{D})) = \sigma_p(\Gamma, X(\mathbb{D})) = \{in : n \in \mathbb{Z}_+\}$, *and for each* $n \geq 0$, $\ker(in - \Gamma) =$ *span*(z^n).
2. *If $\lambda \in \rho(\Gamma)$, then $\mathcal{R}(M_z^m)$ is $R(\lambda, \Gamma)$-invariant for every $m \in \mathbb{Z}_+$, $m > \Im(\lambda)$. In fact, if $h \in \mathcal{R}(M_z^m)$, then*

$$R(\lambda, \Gamma)h(z) = iz^{-\lambda i} \int_0^z \omega^{\lambda i - 1} h(\omega)\, d\omega = iz^m \int_0^1 t^{m + \lambda i - 1} Q^m h(tz)\, dt. \tag{5.10}$$

Proof: Just as before, that T_t is a surjective isometry implies that $\sigma(\Gamma, X(\mathbb{D})) \subseteq i\mathbb{R}$. Let $\lambda \in \mathbb{C}$ and $h \in \mathcal{H}(\mathbb{D})$. Then the equation $(\lambda - \Gamma)f(z) = h(z)$ is equivalent to

$$f'(z) + \frac{i\lambda}{z} f(z) = \frac{i}{z} h(z),$$

or

$$\left(z^{i\lambda} f(z)\right)' = iz^{i\lambda - 1} h(z). \tag{5.11}$$

In particular, $(\lambda - \Gamma)f = 0$ implies that $f(z) = Kz^{-i\lambda}$ for some constant K. Since $z^{-i\lambda} \in \mathcal{H}(\mathbb{D})$ if and only if $-i\lambda \in \mathbb{Z}_+$, it follows that

$$\sigma_p(\Gamma, X(\mathbb{D})) = \sigma_p(\Gamma, \mathcal{H}(\mathbb{D})) = \{in : n \in \mathbb{Z}_+\},$$

with $\ker(in - \Gamma) = \operatorname{span}(z^n)$.

Notice that for $\lambda \in \mathbb{C} \setminus \sigma_p(\Gamma)$ and $f \in \mathcal{H}(\mathbb{D})$, the functions f and $(\lambda - \Gamma)f$ have the same order of zero at 0. Thus $\mathcal{R}(M_z^m, \mathcal{H}(\mathbb{D}))$ is invariant under $(\lambda - \Gamma)$. Fix $\lambda \in \mathbb{C} \setminus \sigma_p(\Gamma)$ and let $m \in \mathbb{Z}_+$ be such that $\Im(\lambda) < m$. Then, for $h(z) = z^m g(z) \in \mathcal{R}(M_z^m, \mathcal{H}(\mathbb{D}))$, the integral

$$i \int_0^z \omega^{\lambda i - 1} h(\omega)\, d\omega = i z^{m + \lambda i} \int_0^1 t^{m + \lambda i - 1} g(tz)\, dt$$

is absolutely convergent. Thus equation (5.11) has unique solution

$$f(z) = i z^m \int_0^1 t^{m + \lambda i - 1} g(tz)\, dt.$$

For every $g \in X(\mathbb{D})$ and t, $0 \leq t < 1$, we have $\|g(tz)\| \leq \|g\|$ since, for every $g \in \mathcal{H}(\mathbb{D})$ and $p > 0$, the function $M_p^p(r, g) := \frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\theta})|^p\, d\theta$ is an increasing function of r, [15, Theorem 1.5]. Minkowski's inequality again implies that for $h \in \mathcal{R}(M_z^m, X(\mathbb{D}))$,

$$\|f\| \leq \int_0^1 t^{m - \Im(\lambda) - 1} \|Q^m h\|\, dt \leq \frac{\|Q\|^m}{m - \Im(\lambda)} \|h\|,$$

and so $(\lambda - \Gamma)|_{\mathcal{R}(M_z^m, X(\mathbb{D}))}$ has bounded resolvent

$$\begin{aligned} R_m(\lambda, \Gamma)h(z) &= i z^m \int_0^1 t^{m - \Im(\lambda) - 1} Q^m h(tz)\, dt \\ &= i z^{-\lambda i} \int_0^z \omega^{\lambda i - 1} h(\omega)\, d\omega. \end{aligned}$$

For every $n \in \mathbb{Z}_+$, the equation $(\lambda - \Gamma)f = z^n$ has unique solution $f(z) = \frac{i}{n + \lambda i} z^n$. If $h \in X(\mathbb{D})$ is arbitrary, write $h(z) = \sum_{n \in \mathbb{Z}_+,\, n < m} a_n z^n + P_m h(z)$. Then

$$R(\lambda, \Gamma)h(z) = \sum_{n \in \mathbb{Z}_+,\, n < m} a_n \frac{i z^n}{n + \lambda i} + R_m(\lambda, \Gamma) P_m h(z) \tag{5.12}$$

is bounded on $X(\mathbb{D})$. Thus $\sigma(\Gamma) = \sigma_p(\Gamma)$. ■

Another special property of the resolvent operator in this case is compactness, as we give in the next result.

Theorem 22

Let $X(\mathbb{D})$ denote either of the spaces $H^p(\mathbb{D})$ or $L^p_a(\mathbb{D}, m_\alpha)$, $\alpha > -1$. Then for every $\lambda \in \rho(\Gamma, X(\mathbb{D}))$, the resolvent $R(\lambda, \Gamma)$ is compact.

Proof: Fix $\lambda \in \rho(\Gamma)$ and let $m \in \mathbb{Z}_+$ be such that $\Im(\lambda) < m$. Since $\mathcal{R}(M_z^m)$ has finite codimension in $X(\mathbb{D})$, it suffices to show that $R_m(\lambda, \Gamma) = R(\lambda, \Gamma)|_{\mathcal{R}(M_z^m)}$ is compact.

If $r > 0$, let $\mathcal{A}(r\mathbb{D})$ be the disc algebra $\mathcal{A}(r\mathbb{D}) = C(r\overline{\mathbb{D}}) \cap \mathcal{H}(r\mathbb{D})$, equipped with the supremum norm, and for each t, $0 \le t < 1$, and $f \in \mathcal{H}(\mathbb{D})$, let $H_t f(z) = f_t(z) = f(tz)$. Then, as in the proof of Theorem 20, for every t, $0 \le t < 1$, H_t is a contraction on $X(\mathbb{D})$.

By equation (5.10), $R_m(\lambda, \Gamma) = iM_z^m \int_0^1 t^{m+\lambda i-1} H_t Q^m \, dt$ with convergence in norm. For each r, $0 < r < 1$, let $C_r = iM_z^m \int_0^r t^{m+\lambda i-1} H_t Q^m \, dt$ on $\mathcal{R}(M_z^m, X(\mathbb{D}))$. Then

$$\|R_m - C_r\| \le \int_r^1 t^{m-\Im(\lambda)-1} \|Q\|^m \, dt = \frac{\|Q\|^m}{m - \Im(\lambda)}(1 - r^{m-\Im(\lambda)}) \to 0$$

as $r \to 1^-$. Choose s so that $1 < s < r^{-1}$. Then $C_r : \mathcal{R}(M_z^m, X(\mathbb{D})) \to \mathcal{R}(M_z^m, X(\mathbb{D}))$ factors through $\mathcal{A}(s\mathbb{D})$. If $\mathbb{B}$ denotes the closed unit ball of $\mathcal{R}(M_z^m, X(\mathbb{D}))$, then the growth conditions given by equations (2.1) and (5.3), imply that for all $f \in \mathbb{B}$ and $z \in s\overline{\mathbb{D}}$,

$$|C_r f(z)| \le \frac{K\, r^{m-\Im(\lambda)}}{(1 - rs)^\gamma (m - \Im(\lambda))},$$

and

$$|(C_r f)'(z)| \le \frac{K\, r^{m-\Im(\lambda)}}{(1 - rs)^{\gamma+1} (m - \Im(\lambda))}.$$

Thus $C_r\mathbb{B}$ is pre-compact in $\mathcal{A}(s\mathbb{D})$ by Arzela-Ascoli. Since $\mathcal{A}(s\mathbb{D})$ is continuously embedded in $X(\mathbb{D})$, it follows that $C_r\mathbb{B}$ is pre-compact in $X(\mathbb{D})$ as well. Thus each C_r is compact in $\mathcal{L}(\mathcal{R}(M_z^m, X(\mathbb{D})))$. It follows that $R_m(\lambda, \Gamma) = \text{(norm)} \lim_{r \to 1^-} C_r$ is compact as well. ∎

Similarly, as in Section 5.1, we end this section by revisiting Example 3 in Chapter 3, which as earlier indicated, corresponds to the rotation group under consideration. We take care of the assumptions of the example, that is; $k = -2$, $g(z) = \psi^{-1}(z) = \frac{z-i}{z+i}$, and $g^{-1}(z) = \psi(z) = \frac{i(1+z)}{1-z}$. Let $u_t(z) = e^{-2it}z$. Then as proved in the Example 3;

$$\begin{aligned}\varphi_t(z) := \frac{z\cos t - \sin t}{z\sin t + \cos t} &= g^{-1}(e^{-2it}g(z)) \\ &= g^{-1} \circ u_t \circ g(z).\end{aligned}$$

Now by definition, $S_{u_t}f(z) = (u_t')^{\gamma} f(u_t(z)) = e^{-2it}f(e^{-2it}z)$. Comparing with the group under consideration $T_t f(z) = e^{ict}f(e^{ikt})$, we see that $c = -2\gamma$ and $k = -2$. Therefore based on the notation developed earlier, the infinitesimal generator of S_{u_t} is $\Gamma_{-2\gamma,-2}$. From Theorems 20 and 21, as well as Proposition 18, we obtain the following consequence,

Corollary 5

For $X(\mathbb{D}) = H^p(\mathbb{D})$ or $L^p_a(\mathbb{D}, m_\alpha)$, let S_{u_t} be the group of isometries on $X(\mathbb{D})$ defined above, and $\Gamma_{-2\gamma,-2}$ be its generator. Then

1. $\Gamma_{-2\gamma,-2}f(z) = i\left(-2\gamma f(z) - 2zf'(z)\right)$ *for every* $f \in X(\mathbb{D})$, *with domain* $\mathcal{D}(\Gamma_{-2\gamma,-2}, X(\mathbb{D})) = \{f \in X(\mathbb{D}) : f' \in X(\mathbb{D})\}$.
2. $\sigma(\Gamma_{-2\gamma,-2}, X(\mathbb{D})) = \sigma_p(\Gamma_{-2\gamma,-2}, X(\mathbb{D})) = \{-2(\gamma+n)i : n \in \mathbb{Z}_+\}$, *and for each* $n \geq 0$, $\ker\left(-2(\gamma+n)i - \Gamma_{-2\gamma,-2}\right) = \text{span}(z^n)$
3. *If* $\mu \in \rho(\Gamma_{-2\gamma,-2})$, *then* $\mathcal{R}(M_z^m)$ *is* $R(\mu, \Gamma_{-2\gamma,-2})$*-invariant for every* $m \in \mathbb{Z}_+$, $m > \Im\left(-(\mu+2\gamma i)/2\right)$. *Moreover, if* $h \in \mathcal{R}(M_z^m)$, *then*

$$R(\mu, \Gamma_{-2\gamma,-2})h(z) = -\frac{i}{2}z^{(\frac{\mu+2i\gamma}{2})i}\int_0^z \omega^{-(\frac{\mu-2i\gamma}{2})i-1}h(\omega)\,d\omega.$$

Proof: Following Proposition 18, $\mu \in \rho(\Gamma_{-2\gamma,-2})$ is equivalent to $-\frac{(\mu+2\gamma i)}{2} \in \rho(\Gamma_{0,1})$. The rest of the proof now follows at once from Theorems 20 and 21, and Proposition 18 as well. ∎

The following proposition details the analysis of this specific group, that is, the group considered in the Example 3.

Proposition 19

Let $X = H^p(\mathbb{U})$ *or* $L^p_a(\mathbb{U}, \mu_\alpha)$, $1 \leq p < \infty$, $\alpha > -1$. *Let* $\varphi_t \in \mathrm{Aut}(\mathbb{U})$ *be given by* $\varphi_t(z) = \frac{z\cos t - \sin t}{z \sin t + \cos t}$, *for all* $t \in \mathbb{R}$, $z \in \mathbb{U}$, *and the corresponding group of isometries on* X *by* $S_{\varphi_t} f(z) := (\varphi_t')^\gamma f(\varphi_t(z))$. *Then*

1. *The infinitesimal generator* Δ *of the group* $S_{\varphi_t} \subset \mathcal{L}(X)$ *is given by*
$$\Delta(h(z)) = -2\gamma z h(z) - (1+z^2)h'(z),$$
with domain $\mathcal{D}(\Delta) = \{h \in X(\mathbb{D}) : 2\gamma(\omega+i)h + (\omega+i)^2 h' \in X\}$.
2. $\sigma_p(\Delta) = \sigma(\Delta) = \{-2(\gamma+n)i : n \in \mathbb{Z}_+\}$, *and for each* $n \geq 0$, $\ker(-2(\gamma+n)i - \Delta) =$ *span*$(S_g z^n)$.
3. *If* $\mu \in \rho(\Delta)$ *and if* $m \in \mathbb{Z}_+$ *is such that* $m > \Im(-(\mu+2i\gamma)/2)$. *Then, if* $h \in \mathcal{R}(M_z^m)$, *we have*
$$R(\mu,\Delta)h(z) = (z-i)^{\frac{\mu+2i\gamma}{2}i}(z+i)^{-(\frac{\mu+2i\gamma}{2}i+2\gamma)} \int_0^z (\omega-i)^{-(\frac{\mu+2i\gamma}{2})i-1} (\omega+i)^{\frac{\mu+2i\gamma}{2}i+2\gamma-1} h(\omega)\, d\omega. \quad (5.13)$$
4. $R(\mu,\Delta)$ *is compact on* $X(\mathbb{D})$.

Proof: Since $\varphi_t = g^{-1} \circ u_t \circ g$, it follows that $S_{\varphi_t} = S_g S_{u_t} S_{g^{-1}} = S_g S_{u_t} S_g^{-1}$. Let Δ be the generator of S_{φ_t} and $\Gamma := \Gamma_{-2\gamma,-2}$ be the generator of S_{u_t}, then as noted before,

$$\Delta = S_g \Gamma S_g^{-1} \text{ with domain } \mathcal{D}(\Delta) = S_g\mathcal{D}(\Gamma).$$

As in the proof of Proposition 17, let $f' \in X(\mathbb{D})$. Then $f \in \mathcal{D}(\Gamma)$ and $h := S_g f$ belongs to $\mathcal{D}(\Delta)$ with $f = S_g^{-1}h$. Then

$$\begin{aligned}\Delta(h(z)) &= S_g\Gamma S_g^{-1}h(z) = S_g \Gamma f(z) = S_g\left(-2\gamma i f(z) - 2iz f'(z)\right)\\ &= -\frac{(2i)^\gamma}{(z+i)^{2\gamma}}\left(2\gamma i f(g(z)) + 2ig(z)f'(g(z))\right).\end{aligned}$$

But $f(z) = S_g^{-1}h(z) = S_{g^{-1}}h(z) = \frac{(2i)^\gamma}{(1-z)^{2\gamma}}h(g^{-1}(z))$, implying that $f(g(z)) = \frac{1}{(2i)^\gamma}(z + i)^{2\gamma}h(z)$. Moreover, $f'(z) = \frac{(2i)^\gamma}{(1-z)^{2\gamma+2}}\left(2\gamma(1-z)h(g^{-1}(z)) + 2ih'(g^{-1}(z))\right)$, implying that

$$f'(g(z)) = \frac{1}{(2i)^{\gamma+1}}(z+i)^{2\gamma+1}\left(2\gamma h(z) + (z+i)h'(z)\right).$$

Therefore,

$$\begin{aligned}\Delta(h(z)) &= -\left(2i\gamma h(z) + 2\gamma(z-i)h(z) + (z-i)(z+i)h'(z)\right)\\ &= -2\gamma z h(z) - (1+z^2)h'(z), \text{ as desired.}\end{aligned}$$

As noted before, the domain of Δ, $\mathcal{D}(\Delta)$ is given by $\mathcal{D}(\Delta) = S_g\mathcal{D}(\Gamma) = \{S_g f : f \in \mathcal{D}(\Gamma)\}$. Now $h \in \mathcal{D}(\Delta) \Leftrightarrow S_g^{-1}h \in \mathcal{D}(\Gamma) \Leftrightarrow (S_{g^{-1}}h)' \in X(\mathbb{D})$. But

$$\begin{aligned}(S_{g^{-1}}h)' &= ((\psi')^\gamma h \circ \psi)'\\ &= \frac{(2i)^\gamma}{(1-z)^{2\gamma}}\left(\frac{2\gamma}{1-\psi^{-1}\circ\psi(z)}h(\psi(z)) + \frac{2i}{1-\psi^{-1}\circ\psi(z)}h'(\psi(z))\right)\\ &= S_{g^{-1}}\left(\frac{2\gamma}{1-\psi^{-1}(\omega)}h(\omega) + \frac{2i}{1-\psi^{-1}(\omega)}h'(\omega)\right).\end{aligned}$$

Therefore,

$$\begin{aligned}h \in \mathcal{D}(\Delta) &\Leftrightarrow S_{g^{-1}}\left(\frac{2\gamma}{1-\psi^{-1}(\omega)}h(\omega) + \frac{2i}{1-\psi^{-1}(\omega)}h'(\omega)\right) \in X(\mathbb{D})\\ &\Leftrightarrow \left(\frac{2\gamma}{1-\psi^{-1}(\omega)}h(\omega) + \frac{2i}{1-\psi^{-1}(\omega)}h'(\omega)\right) \in X\\ &\Leftrightarrow \frac{\omega+i}{2i}\left[2\gamma h(\omega) + (\omega+i)h'(\omega)\right] \in X,\end{aligned}$$

which implies that $\mathcal{D}(\Delta) = \{h \in X(\mathbb{D}) : 2\gamma h(\omega) + (\omega+i)h'(\omega) \in X\}$.

Again, it's clear from Subsection 1.4.3 that,

$$\sigma_p(\Delta) = \sigma_p(\Gamma) = \sigma(\Gamma) = \sigma(\Delta) = \{-2(\gamma+n)i : n \in \mathbb{Z}_+\},$$

but with $\ker(-2(\gamma+n)i-\Delta) = \text{span}(S_g z^n)$ for each $n \geq 0$.

For the resolvents, if $\mu \in \rho(\Delta) = \rho(\Gamma)$, then for $m \in \mathbb{Z}_+$,

$m > \Im\left(-(\mu+2\gamma i)/2\right)$, and if $h \in \mathcal{R}(M_z^m)$, we have $R(\mu,\Delta) = S_g R(\mu,\Gamma) S_g^{-1}$ and so

$$\begin{aligned}
R(\mu,\Delta)h(z) &= S_g\left(-\frac{i}{2} z^{\frac{\mu+2\gamma i}{2} i} \int_0^z \omega^{-(\frac{\mu+2\gamma i}{2})i-1} S_{g^{-1}} h(\omega)\, d\omega\right) \\
&= S_g\left(-\frac{i}{2} z^{\frac{\mu+2\gamma i}{2} i} \int_0^z \omega^{-(\frac{\mu+2\gamma i}{2})i-1} \frac{(2i)^\gamma}{(1-\omega)^{2\gamma}} h(g^{-1}(\omega))\, d\omega\right) \\
&= -\frac{i}{2} \cdot \frac{(2i)^\gamma}{(z+i)^{2\gamma}} (g(z))^{\frac{\mu+2\gamma i}{2} i} \int_0^z (g(\omega))^{-(\frac{\mu+2\gamma i}{2})i-1} \frac{(2i)^\gamma}{(1-g(\omega))^{2\gamma}} h(\omega)\, \frac{dg}{d\omega}\, d\omega \\
&= \left(\frac{z-i}{(z+i)^{2\gamma}}\right)^{\frac{\mu+2i\gamma}{2} i} \int_0^z (\omega-i)^{-(\frac{\mu+2i\gamma}{2})i-1} (\omega+i)^{\frac{\mu+2i\gamma}{2} i+2\gamma-1} h(\omega)\, d\omega.
\end{aligned}$$

Finally, compactness of the resolvent operator $R(\mu,\Delta)$ follows from Theorem 22.

■

5.2.1 Adjoints of the resolvent operator

The duality of Hardy and Bergman spaces of the unit disc $\mathbb{D}$ is well established in literature. Following [21] or [41], for $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\alpha > -1$,

$$(L_a^p(\mathbb{D}, m_\alpha))^* \approx L_a^q(\mathbb{D}, m_\alpha), \tag{5.14}$$

under the pairing

$$\langle f, g\rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}\, dm_\alpha(z) \quad (f \in L_a^p(\mathbb{D}, m_\alpha),\ g \in L_a^q(\mathbb{D}, m_\alpha)),$$

while for the Hardy spaces of the disc $\mathbb{D}$,

$$(H^p(\mathbb{D}))^* \approx H^q(\mathbb{D}), \tag{5.15}$$

via

$$\langle f, g\rangle = \int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\, d\theta \quad (f \in H^p(\mathbb{D}),\ g \in H^q(\mathbb{D})).$$

For $p = 1$, it is proved in [41, Theorems 5.3 and 5.15] that dual and the predual of the Bergman space $L^1_a(\mathbb{D}, m_\alpha)$ are respectively given by

$$(L^1_a(\mathbb{D}, m_\alpha))^* \approx \mathcal{B}_\infty(\mathbb{D}), \quad \text{and} \quad (\mathcal{B}_{\infty,0}(\mathbb{D}))^* \approx L^1_a(\mathbb{D}, m_\alpha), \tag{5.16}$$

with the usual integral pairing similar to the above. Under the above pairings, the adjoint operator is conjugate linear.

Recall that for the rotation group, we considered $\varphi_t(z) = e^{ikt}z$ but with the corresponding group of isometries on $X(\mathbb{D})$ given by $T_tf(z) = e^{ict}f(e^{ikt}z)$ for every $f \in X(\mathbb{D})$.

Now, let $k = 1$ so that $\varphi_t(z) = e^{it}z$. Therefore by definition, $T_tf := S_{\varphi_t}f = (\varphi_t')^\gamma f(\varphi_t)$, we have $T_tf(z) = S_{\varphi_t}f(z) = e^{i\gamma t}f(e^{it}z)$, that is, $c = \gamma$. Using the notation adopted earlier, $\Gamma_{\gamma,1}$ is the generator of this group, and we denote its resolvent operator $R(\lambda, \Gamma_{\gamma,1})$ by R_λ. Then clearly, R_λ is bounded on $X(\mathbb{D}) = H^p(\mathbb{D})$ or $L^p_a(\mathbb{D}, m_\alpha)$.

We wish to study the adjoint R^*_λ of R_λ. Just as is the case in Section 5.1 above, we let $1 < p, q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. We first obtain the adjoint T^*_t of the group T_t in the following result,

Theorem 23

Let $X(\mathbb{D}) = H^p(\mathbb{D})$ or $L^p_a(\mathbb{D}, m_\alpha)$, $1 < p < \infty$. Let $\gamma = \frac{\alpha+2}{p}$ and $T_tf(z) = e^{i\gamma t}f(e^{it}z)$ for every $f \in X(\mathbb{D})$, and define $T_{-t}g(z) = e^{-i\gamma t}f(e^{-it}z)$ for all $g \in X(\mathbb{D})^*$. Then $(T_t)_{t\in\mathbb{R}}$ and $(T_{-t})_{t\in\mathbb{R}}$ are adjoints of each other; that is, $T^*_t = T_{-t}$.

Proof: If $X(\mathbb{D}) = L^p_a(\mathbb{D}, m_\alpha)$, then $X^* = L^q_a(\mathbb{D}, m_\alpha)$ and for all $f \in X(\mathbb{D})$, $g \in X(\mathbb{D})^*$, we have,

$$\begin{aligned}\langle T_t f, g\rangle &= \int_{\mathbb{D}} e^{i\gamma t} f(e^{it}z)\overline{g(z)}\, dm_\alpha(z) = \int_{\mathbb{D}} e^{i\gamma t} f(e^{it}z)\overline{g(z)}(1-|z|^2)^\alpha\, dA(z)\\ &= \int_{\mathbb{D}} f(\omega)\overline{e^{-i\gamma t} g(e^{-it}\omega)}\, dm_\alpha(\omega) = \langle f, T_{-t} g\rangle, \quad \text{as desired.}\end{aligned}$$

If $X(\mathbb{D}) = H^p(\mathbb{D})$, then $X^* = H^q(\mathbb{D})$ and for all $f \in X(\mathbb{D})$, $g \in X(\mathbb{D})^*$, we have

$$\begin{aligned}\langle T_t f, g\rangle &= \int_0^{2\pi} e^{i\gamma t} f(e^{it}e^{i\theta})\overline{g(e^{i\theta})}\, d\theta = \int_0^{2\theta} f(e^{i(t+\theta)})\overline{e^{-i\gamma t} g(e^{i\theta})}\, d\theta\\ &= \int_0^{2\theta} f(e^{i\omega})\overline{e^{-i\gamma t} g(e^{i(\omega-t)})}\, d\omega = \langle f, T_t g\rangle.\end{aligned}$$

∎

Since $X(\mathbb{D}) = H^p(\mathbb{D})$ or $L^p_a(\mathbb{D}, m_\alpha)$, $1 < p < \infty$, is reflexive, it follows that if $(T_t)_{t\in\mathbb{R}}$ has generator $\Gamma_{\gamma,1} f(z) = i(\gamma f(z) + z f'(z))$ for $f \in X(\mathbb{D})$, then $\Gamma^*_{\gamma,1} = -\Gamma_{\gamma,1}$; that is, $\Gamma^*_{\gamma,1} f(z) = -i(\gamma f(z) + z f'(z)),\ \forall\, g \in X(\mathbb{D})^*$ and $R(\lambda, \Gamma_{\gamma,1})^* = R(\bar{\lambda}, \Gamma^*_{\gamma,1})$.

Theorem 24

Let $X(\mathbb{D})$ be one of the spaces $H^p(\mathbb{D})$ or $L^p_a(\mathbb{D}, m_\alpha)$, $1 < p < \infty$, and let q be conjugate to p. If $\lambda \in \rho(\Gamma_{\gamma,1}, X(\mathbb{D}))$, then

$$R^*_\lambda = -R_{-\bar{\lambda}}.$$

Proof: If $\lambda \in \rho(\Gamma_{\gamma,1}, X(\mathbb{D}))$, then

$$R^*_\lambda = (R(\lambda, \Gamma_{\gamma,1}))^* = R(\bar{\lambda}, \Gamma^*_{\gamma,1}) = R(\bar{\lambda}, -\Gamma_{\gamma,1})$$

$$= -R(-\bar{\lambda}, \Gamma_{\gamma,1}) = -R_{-\bar{\lambda}}, \quad \text{as claimed.}$$

∎

For $p = 1$, the space $L_a^1(\mathbb{D}, m_\alpha)$ is non-reflexive and as given by equation (5.16), its predual is $\mathcal{B}_{\infty,0}(\mathbb{D})$. Thus $T_t f(z) = (e^{it})^{\alpha+2} f(e^{it}z)$ and for all $g \in \mathcal{B}_{\infty,0}$, we have

$$\begin{aligned}
\langle g, T_t f \rangle &= \int_{\mathbb{D}} g(z)\overline{(e^{it})^{\alpha+2} f(e^{it}z)}\, dm_\alpha(z) \\
&= \int_{\mathbb{D}} (e^{-it})^{\alpha+2} g(e^{-it}(e^{it}z))\overline{f(e^{it}z)}(1 - |e^{it}z|^2)^\alpha \, dA(z) \\
&= \int_{\mathbb{D}} T_{-t} g(\omega)\overline{f(\omega)}\, dm_\alpha(\omega) = \langle T_{-t} g, f \rangle.
\end{aligned}$$

Define $S_t g(z) := T_{-t} g(z) = (e^{-it})^{\alpha+2} g(e^{-it}z)$ for every $g \in \mathcal{B}_{\infty,0}(\mathbb{D})$. We then give the following theorem,

Theorem 25

1. *$(S_t)_{t\in\mathbb{R}}$ is a strongly continuous group of isometries on $\mathcal{B}_{\infty,0}(\mathbb{D})$.*
2. *The infinitesimal generator Δ of the group $(S_t)_{t\in\mathbb{R}}$ is $\Delta f(z) = -i(\gamma f(z) + z f'(z))$, where $\gamma = \alpha + 2$ with domain $\mathcal{D}(\Delta) = \{f \in \mathcal{B}_{\infty,0}(\mathbb{D}) : z f' \in \mathcal{B}_{\infty,0}(\mathbb{D})\}$.*

Proof: It's clear that $(S_t)_{t\in R}$ is a group. Indeed, if $f \in \mathcal{B}_{\infty,0}(\mathbb{D})$, then $S_0 f = f$ and $S_t S_s f(z) = S_{t+s} f(z)$. Now, by a simple change of variables,

$$\begin{aligned}
\|S_t f\|_{\mathcal{B}_\infty(\mathbb{D})} &= |S_t f(0)| + \sup_{z\in\mathbb{D}} (1 - |z|^2)\, |(S_t f)'(z)| \\
&= |e^{-i(\alpha+2)t} f(0)| + \sup_{z\in\mathbb{D}} (1 - |z|^2)\left| e^{-i(\alpha+2)t} e^{-it} f'(e^{-it}z)\right| \\
&= |f(0)| + \sup_{\omega\in\mathbb{D}} (1 - |\omega|^2)|f'(\omega)| = \|f\|_{\mathcal{B}_\infty(\mathbb{D})}.
\end{aligned}$$

To prove strong continuity, we apply density of polynomials in $\mathcal{B}_{\infty,0}(\mathbb{D})$. It suffices to show that for $(z^n)_{n\geq 1}$, $\lim_{t\to 0} \|S_t z^n - z^n\|_{\mathcal{B}_\infty} = 0$. Now $S_t z^n - z^n = (e^{-i(\alpha+2+n)t} - 1) z^n$, so that

$$\sup_{z\in\mathbb{D}} (1 - |z|^2)\, |(S_t z^n - z^n)'| = \sup_{z\in\mathbb{D}} (1 - |z|^2) n \left|(e^{-i(\alpha+2+n)t} - 1) z^{n-1}\right| \to 0 \text{ as } t \to 0,$$

which completes the proof of (1).

For (2), the generator Δ of the group S_t is given by

$$\Delta f(z) = \frac{\partial}{\partial t} S_t f(z)\bigg|_{t=0} = -i(\gamma f(z) + z f'(z)), \quad \text{where } \gamma = \alpha + 2,$$

and so $\mathcal{D}(\Delta) \subseteq \{f \in \mathcal{B}_{\infty,0}(\mathbb{D}) : zf' \in \mathcal{B}_{\infty,0}(\mathbb{D})\}$. Now, let $F(z) = -i(\gamma f(z) + z f'(z))$, then

$$\begin{aligned} \frac{S_t f - f}{t} &= \frac{1}{t}\int_0^t \frac{\partial}{\partial s}\left(S_s f(z)\right) ds \\ &= \frac{1}{t}\int_0^t \left(-i\gamma e^{-i\gamma s} f(e^{-is} z) - i z e^{-i\gamma s} e^{-is} f'(e^{-is} z)\right) ds \\ &= \frac{1}{t}\int_0^t T_s F(z)\, ds \;\to\; F(z) \text{ as } t \;\to\; 0. \end{aligned}$$

Thus $\mathcal{D}(\Delta) \supseteq \{f \in \mathcal{B}_{\infty,0}(\mathbb{D}) : zf' \in \mathcal{B}_{\infty,0}(\mathbb{D})\}$. ■

CHAPTER 6

CONCLUSIONS AND FUTURE DIRECTIONS

6.1 Conclusions

In this dissertation, we have fully characterized the one-parameter groups of automorphisms of the upper half-plane into three types. We have also obtained integral operators as concrete representations of the resolvents of generators of all the groups of weighted composition operators associated with the automorphism groups. Further, we have obtained the spectral properties of the generators of the groups and the resolvents of the generators acting on Banach spaces of analytic functions; specifically, the Hardy and weighted Bergman spaces of the upper half-plane, and in some instances of the disc.

6.2 Future directions

As for the future work, we shall focus on the following open problems that have become apparent from this study:

First is the extension of this study to other spaces of analytic functions of the upper half-plane. For instance, the Dirichlet spaces, Bloch spaces, and spaces of bounded and vanishing mean oscillation. This would require a deeper understanding of the structure of composition operators on such spaces.

Equally of interest are the duality problems for both $L^1_a(\mathbb{U}, \mu_\alpha)$ and $H^1(\mathbb{U})$. The duality properties of the corresponding spaces on the disc, that is, $L^1_a(\mathbb{D}, m_\alpha)$ and $H^1(\mathbb{D})$, have been determined and well documented in literature. For the upper half-plane $\mathbb{U}$, Coifman and Rochberg [9] determined the dual space of the unweighted Bergman space $L^1_a(\mu_0)$, while Kang [24] determined the dual space for the weighted case. The study of predual spaces can be considered. On the other hand, for Hardy spaces, Fefferman and Stein [19] established the dual space of $H^1(\mathbb{R}^n)$. It will be interesting to specifically establish this Fefferman's result for the half-plane as well as the predual space of $H^1(\mathbb{U})$, which just as in the case of the disc, may most probably be given in terms of the Sarason's [33] space of vanishing mean oscillations of the half-plane.

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