Near field phenomena in dipole radiation

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In this dissertation we have studied near-field phenomena in dipole radiation.

We have studied first the energy flow patterns of the radiation emitted by an electric dipole located in between parallel mirrors. The field lines of the Poynting vector have intricate structures, including many singularities and vortices. For a dipole parallel to the mirror surfaces, vortices appear close to the dipole. Vortices are located where the magnetic field vanishes. Also, a radiating electric dipole near the joint of two orthogonal mirrors is considered, and also here we find numerous singularities and vortices in the energy flow patterns. We have also studied the current density in the mirrors.

Next we have studied the reflection of radiation by and the transmission of radiation through an interface with an $\varepsilon$-near-zero (ENZ) material. For $p$ polarization, we find that the reflection coefficient is -1, and the transmission coefficient is zero for all angles of incidence. The transmitted electric field is evanescent and circularly polarized. The transmitted magnetic field is identically zero. For $s$ polarization, the transmitted electric field is $s$ polarized and the transmitted magnetic field is circularly polarized.
The next topic was the study of the force exerted on the dipole by its own reflected field near an ENZ interface. We found that, under certain circumstances, it could be possible that the dipole would levitate in its reflected field. This levitation is brought about by evanescent reflected waves.

Finally, power emission by an electric dipole near an interface was considered. We have derived expressions for the emitted power crossing an interface. The power splits in contributions from traveling and evanescent incident waves. We found that for an ENZ interface, only evanescent dipole waves penetrate the material, but there is no net power flow into the material.
DEDICATION

For my academic advisor Dr. Arnoldus who guided me in this process and the committee who kept me on track.
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CHAPTER I

ENERGY FLOW OF ELECTRIC DIPOLE RADIATION IN BETWEEN PARALLEL MIRRORS

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We have studied the energy flow patterns of the radiation emitted by an electric dipole located in between parallel mirrors. It appears that the field lines of the Poynting vector (the flow lines of energy) can have very intricate structures, including many singularities and vortices. The flow line patterns depend on the distance between the mirrors, the distance of the dipole to one of the mirrors and the angle of oscillation of the dipole moment with respect to the normal of the mirror surfaces. Already for the simplest case of a dipole moment oscillating perpendicular to the mirrors, singularities appear at regular intervals along the direction of propagation (parallel to the mirrors). For a parallel dipole, vortices appear in the neighborhood of the dipole. For a dipole oscillating under a finite angle with the surface normal, the radiating tends to swirl around the dipole before travelling off parallel to the mirrors. For relatively large mirror separations, vortices appear in the pattern. When the dipole is off-centered with respect to the midway point between the mirrors, the flow line structure becomes even more complicated, with numerous vortices in the pattern, and tiny loops near the dipole. We have also investigated the locations of the vortices and...
singularities, and these can be found without any specific knowledge about the flow lines. This provides an independent means of studying the propagation of dipole radiation between mirrors.

1.1 Introduction

An atom or molecule in an excited electronic state will decay spontaneously to the ground state, and at the same time a fluorescent photon is emitted. Assuming a steady excitation of the particle, for instance by a laser beam, this leads to a steady emission of photons. The emission rate, multiplied by the energy of a photon, is the emitted power, and the inverse of the emission rate is the lifetime of the excited state. We shall assume that the particle can be represented by an oscillating electric dipole moment. When the particle is located close to an interface with a dielectric material or a mirror, the photon emission rate is altered due to the fact that the reflected radiation partially travels back to the particle and the reflected electric field at the location of the dipole alters the emission rate \[1\]. Of particular interest is the dependence of the emission rate on the distance between the dipole and the interface. The dependence of the emission rate on the various parameters has been studied by numerous authors, both theoretically and experimentally \[2–10\]. Also of interest is the modification of the far-field radiation pattern, as it is modified by interference between directly emitted and reflected light \[11, 12\]. An obvious generalization of the case of a dipole near an interface is the situation where the emitter is located in between two parallel interfaces \[13–18\]. When the two interfaces are mirrors, a relatively simple expression can be derived for the emission rate. It follows theoretically that spontaneous emission is ‘turned off’ entirely when the distance between the mirrors is less than half a wavelength of the radiation, and when the dipole moment is oriented parallel to the mirror surfaces. This has been observed experimentally for Rydberg atoms in between parallel mirrors \[19\].
When a small particle emits radiation near an interface, the presence of the interface alters the decay rate due to reflected radiation which comes back to the particle. A simultaneous phenomenon is that the reflected light and the direct light interfere, and this is responsible for constructive and destructive interference, leading to lobes in the power per unit solid angle in the far field. In the near field, however, this interference leads to complicated energy flow patterns in the neighborhood of the particle. For a single mirror, numerous singularities and vortices appear [20] and the flow lines of energy are far from trivial. From a larger view, it appears that four strings containing small vortices appear to come out of the location of the dipole [21]. An interesting phenomenon occurs when dipole radiation passes through an interface with a dielectric [22, 23]. When the medium is thicker than the embedding medium of the dipole, radiation passes through more or less in straight lines, similar to optical rays. When the medium is thinner, however, some radiation that transmits into the medium turns around and passes through the interface again. Then, it turns around again and so on. This leads to an oscillating energy flow back and forth through the interface, and at each crossing a vortex appears.

1.2 Dipole in between mirrors

We shall consider an electric dipole located in between parallel mirrors, as shown in Figure 1.1. The dipole is located on the z axis, a distance $H$ above the lower mirror. The surface of the lower mirror is the $xy$ plane and the surface of the second mirror is the $z = D$ plane. The dipole moment $\mathbf{d}(t)$ oscillates harmonically with angular frequency $\omega$:

$$\mathbf{d}(t) = \text{Re} \left[ \mathbf{d} e^{-i\omega t} \right] , \quad (1.1)$$
with \( \mathbf{d} \) the complex amplitude. We set

\[
\mathbf{d} = d_o \hat{\mathbf{u}}_0 ,
\]  

(1.2)

with \( d_o > 0 \) and \( \hat{\mathbf{u}}_0^* \cdot \hat{\mathbf{u}}_0 = 1 \). Here, the unit polarization vector \( \hat{\mathbf{u}}_0 \) may be complex (as for circular or elliptical polarization). The electric field \( \mathbf{E}(\mathbf{r}, t) \) in between the mirrors is

\[
\mathbf{E}(\mathbf{r}, t) = \text{Re}[\mathbf{E}(\mathbf{r})e^{-i\omega t}] ,
\]  

(1.3)

with \( \mathbf{E}(\mathbf{r}) \) the complex amplitude and the magnetic field \( \mathbf{B}(\mathbf{r}, t) \) is expressed similarly. Here, \( \mathbf{r} \) is the position vector of the field point of interest. For the radiation emitted by the dipole, it is convenient to introduce the position vector of the field point with respect to the location of the dipole as follows:

\[
\mathbf{r}_0 = \mathbf{r} - H\mathbf{e}_z ,
\]  

(1.4)

and this vector is shown in Figure 1.1. With the help of the wave number \( k_o = \omega / c \) we introduce dimensionless variables. The dimensionless position vector of the field point is \( \mathbf{q} = k_o \mathbf{r} \) and similarly \( \mathbf{q}_0 = k_o \mathbf{r}_0 \). The dimensionless complex amplitudes \( \mathbf{e}(\mathbf{r}) \) and \( \mathbf{b}(\mathbf{r}) \) of the electric and magnetic fields, respectively, are defined as follows:

\[
\mathbf{E}(\mathbf{r}) = \zeta \mathbf{e}(\mathbf{r}) ,
\]  

(1.5)
\[ \mathbf{B}(\mathbf{r}) = \frac{\zeta}{c} \mathbf{b}(\mathbf{r}) , \]  

(1.6)

with

\[ \zeta = \frac{k_0^3 d_0}{4\pi \varepsilon_0} . \]  

(1.7)

For the fields emitted by the dipole we then have [24]

\[ \mathbf{e}_0(\mathbf{r}) = \left\{ \hat{\mathbf{u}}_0 - (\hat{\mathbf{q}}_0 \cdot \hat{\mathbf{u}}_0)\hat{\mathbf{q}}_0 + [\hat{\mathbf{u}}_0 - 3(\hat{\mathbf{q}}_0 \cdot \hat{\mathbf{u}}_0)\hat{\mathbf{q}}_0] \frac{i}{q_0} \left( 1 + \frac{i}{q_0} \right) \} \frac{e^{i q_0}}{q_0} , \]  

(1.8)

\[ \mathbf{b}_0(\mathbf{r}) = (\hat{\mathbf{q}}_0 \times \hat{\mathbf{u}}_0) \left( 1 + \frac{i}{q_0} \right) \frac{e^{i q_0}}{q_0} . \]  

(1.9)

The subscripts ‘0’ everywhere are for later purpose.
1.3 The image system

The radiation emitted by the dipole bounces off the mirrors and multiple reflections lead to complicated interference patterns. This system can be analyzed most easily with the method of images. In between the mirrors, the dipole field is a solution of Maxwell equations. At the mirror surfaces we must have that the parallel component of the electric field vanishes, and the perpendicular component of the magnetic field must be zero. To this end, mirror images are introduced outside the region $0 < z < D$, such that the total fields at the mirror surfaces satisfy the boundary conditions. For a single (bottom) mirror, this image is located at a distance $H$ below the mirror and its dipole moment has its parallel component reversed. So, if we write

\[ \mathbf{S}(\mathbf{r}) \]

\[ \mathbf{r}_0 \]

\[ \hat{\mathbf{u}}_0 \]

\[ \gamma \]

\[ H \]

\[ r \]

\[ r_0 \]

Figure 1.1 The dipole is located on the $z$ axis, a distance $H$ above the lower mirror.

The vector $\mathbf{r}_0$ represents the field point $\mathbf{r}$, but measured from the location of the dipole. The Poynting vector at the field point is indicated by vector $\mathbf{S}(\mathbf{r})$. For a linear dipole, oscillating in the $yz$ plane, the unit vector $\hat{\mathbf{u}}_0$ makes an angle $\gamma$ with the positive $z$ axis.
then the mirror dipole has polarization vector

\[ \hat{\mathbf{u}}_0 = (\hat{\mathbf{u}}_0)_\perp + (\hat{\mathbf{u}}_0)_\parallel, \]

(1.10)

It can then be verified that the total fields satisfy the boundary conditions.

For the case of two mirrors, the dipole has an image in both mirrors. But then, the image of the dipole in the lower mirror produces a field at the position of the upper mirror. Therefore, we need an image of the first image with respect to the upper mirror to compensate for this. This continues indefinitely, leading to an infinite array of images on the \( z \) axis. We shall number the images with \( m \), and such that above the top mirror we have \( m > 0 \) and below the bottom mirror we have \( m < 0 \).
Figure 1.2  The diagram shows the locations of the mirror images
The value \( m = 0 \) is taken to correspond to the dipole itself. It then follows by inspection that the \( m^{th} \) image is located at

\[
z_m = \left(m + \frac{1}{2}\right)D + (-1)^m \left(H - \frac{1}{2}D\right).
\]  

(1.12)

The location of the images is illustrated in Figure 1.2. Equation (1.12) can be written as follows:

\[
z_m = \begin{cases} 
  mD + H, & \text{if } m \text{ even} \\
  (m + 1)D - H, & \text{if } m \text{ odd}
\end{cases}.
\]  

(1.13)

The figure shows that the \( m^{th} \) image is located in between the planes \( z = mD \) and \( z = (m + 1)D \). We also see that images with \( m \) even have polarization vector \( \hat{u}_0 \) and images with \( m \) odd have \( \hat{u}_0^{mi} \) as polarization vector. This can be combined as follows:

\[
\hat{u}_m = (\hat{u}_0)_{\perp} + (-1)^m (\hat{u}_0)_{\parallel}.
\]

(1.14)

### 1.4 Fields and Poynting vector

Each image dipole radiates an electric and a magnetic field, similar to Equations (1.8) and (1.9) for the dipole. For the fields by the \( m^{th} \) image, we replace \( \hat{u}_0 \) by \( \hat{u}_m \) from Equation (1.14) and the dimensionless position vector \( q_0 \) of the field point is replaced by

\[
q_m = q - \bar{z}_m e_z,
\]

(1.15)
with $\bar{z}_m = k_0 z_m$. The dimensionless electric and magnetic fields are $e_m(\mathbf{r})$ and $b_m(\mathbf{r})$, respectively, and the total fields at field point $\mathbf{r}$ are

$$e(\mathbf{r}) = \sum_{m=\infty}^{\infty} e_m(\mathbf{r}) , \quad (1.16)$$

$$b(\mathbf{r}) = \sum_{m=\infty}^{\infty} b_m(\mathbf{r}) . \quad (1.17)$$

Energy in between the mirrors flows along the field lines of the Poynting vector $\mathbf{S}(\mathbf{r},t)$. For time-harmonic fields, we consider the time-averaged Poynting vector:

$$\mathbf{S}(\mathbf{r}) = \frac{1}{2\mu_o} \text{Re}[\mathbf{E}(\mathbf{r})^* \times \mathbf{B}(\mathbf{r})] , \quad (1.18)$$

which is time-independent. We split off a factor

$$\mathbf{S}(\mathbf{r}) = \frac{\xi^2}{2\mu_o c} \mathbf{\sigma}(\mathbf{r}) , \quad (1.19)$$

so that

$$\mathbf{\sigma}(\mathbf{r}) = \text{Re}[\mathbf{e}(\mathbf{r})^* \times \mathbf{b}(\mathbf{r})] . \quad (1.20)$$

With $\mathbf{e}(\mathbf{r})$ and $\mathbf{b}(\mathbf{r})$ from Equations (1.16) and (1.17) substituted, we see that we get cross-terms between all the $m$ values. Vector $\mathbf{\sigma}(\mathbf{r})$ depends only on $\mathbf{q}$ (and not $\mathbf{r}$), so we shall write $\mathbf{\sigma}(\mathbf{q})$. 
Equation (1.20) determines the Poynting vector at a field point \( \mathbf{q} \). Let \( \mathbf{q}(u) \) be a parametrization of a field line through a given initial point \((\bar{x}_0, \bar{y}_0, \bar{z}_0)\), with \( \bar{x}_0 = k_0 x_0 \), etc. The curves \( \mathbf{q}(u) \) are then the solution of

\[
\frac{d}{du} \mathbf{q}(u) = f(\mathbf{q}) \sigma[\mathbf{q}(u)] .
\]

(1.21)

Here, \( f(\mathbf{q}) \) is an arbitrary positive function of \( \mathbf{q} \), which can be selected for convenience or numerical stability. A good choice seems to be \( f(\mathbf{q}) = \mathbf{q}_0^5 \). We use Mathematica to solve Equation (1.21) (which is actually a set of three equations when written out in Cartesian coordinates).

The sums over \( m \) in Equations (1.16) and (1.17) obviously needs to be truncated with a certain maximum \( M \) \((-M < m < M, \) so \( 2M + 1 \) dipoles). This \( M \) is determined by considering the Poynting vector near the two mirrors. At a mirror surface, the Poynting vector must be parallel to the surface (no energy flows through the mirrors). Only for a sufficient large value of \( M \) will this be the case and we find that values of \( M \) of about 100 are usually necessary for convergence.

1.5 Linear dipole

The field lines of the Poynting vector are in general 3D curves, which makes it very difficult to visualize energy flow line patterns. A great simplification arises if we assume that the dipole oscillates linearly, as in Figure 1.1. The direction of oscillation is specified by the angle \( \gamma \) with the positive \( z \) axis and we take the plane of oscillation as the \( yz \) plane. We then have

\[
\hat{\mathbf{u}}_0 = \mathbf{e}_z \cos \gamma + \mathbf{e}_y \sin \gamma ,
\]

(1.22)

and for the images we obtain
\[ \mathbf{u}_m = e_z \cos \gamma + e_y (-1)^m \sin \gamma . \] 

(1.23)

Let us now consider a field point in the \( yz \) plane. It follows from Equation (1.8) (with \( 0 \to m \)) that \( \mathbf{e}_m (\mathbf{q}) \) is in the \( yz \) plane and we see from Equation (1.9) that \( \mathbf{b}_m (\mathbf{q}) \) is along the \( x \) axis. Therefore, the Poynting vector \( \mathbf{\sigma} (\mathbf{q}) \) is in the \( yz \) plane. Consequently, if a field line of the Poynting vector goes through a point in the \( yz \) plane, then the entire field line lies in the \( yz \) plane. Similarly, it is easy to verify that the field line pattern is reflection symmetric in the \( yz \) plane, so we only need to consider field lines in the region \( x \geq 0 \). The \( yz \) plane is a symmetry plane, and field lines in this plane are 2D curves.

1.6 Vertical dipole

In the graphs to follow, we use dimensionless coordinates \( \bar{y} = k_o y \) and \( \bar{z} = k_o z \) for points in the \( yz \) plane and we introduce \( \delta = k_o D \) for the dimensionless distance between the mirrors and \( h = k_o H \) for the dimensionless distance between the dipole and the lower mirror. We shall first consider the case \( \gamma = 0 \), corresponding to a dipole oscillating along the \( z \) axis, and with the dipole located midway between the mirrors. For a linear dipole in free space, the field lines are straight, coming out of the dipole and running to infinity. We see from Figure 1.3 that close to the dipole the field lines indeed come out of the dipole, and almost as straight lines. At this close distance it seems that the mirrors have no effect yet. When the field lines approach the mirrors, they bend and then run off to infinity. For \( \gamma = 0 \), the system is rotation symmetric around the \( \bar{z} \) axis, so there are no additional features off the \( \bar{y}\bar{z} \) plane.
The flow lines in Figure 1.3 are what one would expect for the energy flow in between mirrors when the flow lines come out of the dipole. Figure 1.4 shows an extension to the right of the flow pattern in Figure 1.3. We see that, instead of smoothly running off to infinity, two singularities appear near the midway plane between the mirrors. At a singularity, the Poynting vector vanishes and such singularities are indicated by small circles. Figure 1.5 shows a further extension of the graph, and now we notice that two singularities appear close to the mirror surfaces. Since for a vertical dipole the field line picture is rotation symmetric around the \( \bar{z} \) axis, these singularities are singular circles around the \( \bar{z} \) axis. We have verified numerically that this picture continues to the right indefinitely. After about 8 units further, two singularities appear along the midway plane, then again 8 units further two near the mirrors and so on. The pattern with the sequences of singularities in Figures 1.3–1.5 seems to be universal for a vertical dipole located at the midway point. The locations of the singularities in the horizontal direction depend on \( \delta \). The larger the separation between the mirrors, the further apart the singularities are. However, it seems that there are always two singularities near the midway plane and two singularities near the mirrors. The number of singularities does not seem to increase with the mirror separation. When the mirror separation is less than about \( \delta = \pi \), there are no singularities, and the radiation flows along almost straight lines to infinity. Since \( 2\pi \) corresponds to an optical wavelength, we conclude that the mirror separation must be at least half a wavelength for the singular circles in the flow pattern to appear.
Figure 1.3  The figure shows field lines of the Poynting vector in the $\tilde{y}\tilde{z}$ plane for a vertical dipole.

In this and other figures, we shall take $\tilde{z}$ as up and $\tilde{y}$ to the right. The dipole moment vector $\mathbf{u}_0$ is represented by an arrow (not to scale). The distance between the mirrors is $\delta = 8$ and the locations of the mirrors are indicated by fat black lines. The dipole is located midway between the mirrors, so at $h = 4$.

Figure 1.4  This figure is an extension to the right of Figure 1.3. The white circles are singularities.
1.7 **Horizontal dipole**

We now consider a horizontal dipole ($\gamma = \pi / 2$) located midway between the mirrors. Figure 1.6 shows the energy flow picture for $\delta = 4$. Energy is emitted by the dipole in the up and down directions and the flow lines bend at the mirror surfaces, just like in Figure 1.3. For a dipole in free space, no energy is emitted along the dipole axis and the same appears to hold here. The line $\bar{z} = 2$ is a singular line, with no energy flow along this line (indicated by a dashed line in the figure). Radiation emitted upward and downward changes direction at the mirror surfaces, after which the flow lines bend to the dipole axis $\bar{z} = 2$ and the flow lines end at this axis. For a dipole in free space, the field lines are straight, but due to the presence of the mirrors here, all field lines bend to the dipole axis and end there. This is also clear from Figure 1.7, which is an extension to the right of Figure 1.6.
When the distance between the mirrors is increased, four vortices appear near the location of the dipole. This is shown in Figure 1.8. Not all field lines end at the dipole axis, as for smaller $\delta$. Close to the dipole, field lines start at the dipole axis and then swirl into the vortices, where they end. Further out, the field lines again bend towards the axis, as in Figure 1.7. In the transition region, there are two singularities on the dipole axis, which are indicated by white circles in the figure. For even larger $\delta$, a similar structure as in Figures 1.4 and 1.5 appears, where there are sets of singularities alternating in location.

![Diagram of field lines](image.png)

Figure 1.6  Shown are the field lines of energy flow for a horizontal dipole midway between the mirrors, for $\delta = 4$. 

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Figure 1.7  This figure is an extension to the right of Figure 1.6.

Figure 1.8  The figure shows the energy flow lines for $\delta = 8$, $h = 4$ and $\gamma = \pi / 2$.

The black circles are at the centers of vortices and the white circles are other singularities.
1.8 Dipole under an angle

We now consider the effect of an angle $\gamma$ between the $\bar{z}$ axis and the direction of oscillation $\bar{u}_0$ of the dipole. A typical flow line pattern for small $\delta$ is shown in Figure 1.9. Most of the radiation is emitted perpendicular to the dipole axis, as in free space. We see that field lines coming out of the dipole swing around the dipole and then run off to infinity. Some field lines make half a swing, as for curve $a$, and some make a full swing, as for curve $b$. For larger values of the mirror separation, vortices appear along the dipole axis. This is illustrated in Figure 1.10. Two large vortices form in the top-right and the bottom-left of the picture, and in between the large vortices and the dipole two small vortices appear. An enlargement of the top right vortices is shown in Figure 1.11. When we increase $\delta$ even further, more vortices appear approximately along the dipole axis.

Figure 1.9 The dipole oscillates under 45° with the $\bar{z}$ axis, and we have $\delta = 2$. 
Figure 1.10  Field lines for a dipole oscillating under $45^\circ$ with the $\hat{z}$ axis, and for $\delta = 8$.

Figure 1.11  The figure shows an enlargement of the top-right vortex in Figure 1.10.
The black circles are at the centers of the vortices and the white circles are other singularities.
1.9 Energy flow near a vortex

The energy flow lines near the small vortex in Figure 1.11 end at the center of the vortex and the flow line (only one shown) near the large vortex comes out of the center of the vortex and then runs off to the left. At the center of each vortex is a singularity, indicated by small circles, and three more singularities can be seen in the picture. It seems that the small vortex is an energy sink and the large vortex is an energy source. Since the flow line pattern is time independent, this can obviously not be the case. Energy cannot pile up at the center of the small vortex, and no energy is created at the center of the large vortex. From a mathematical point of view, the divergence of the Poynting vector is zero in between the mirrors, except at the location of the dipole. There are no energy sinks or sources other than the dipole.

We only draw field lines in the symmetry plane in order to avoid cumbersome 3D visualizations. This gives a good impression of the behavior of the energy flow in most cases. However, the field line patterns shown are part of larger, 3D pictures. This becomes most relevant when considering the energy flow near a vortex. Figure 1.12 shows a 3D field line in the neighborhood of the large vortex in Figure 1.11. The field line’s initial point is taken as $(\bar{x}_o, \bar{y}_o, \bar{z}_o) = (0.02, 3, 7)$, so slightly in front of the $\bar{y}\bar{z}$ plane. The field line is spiralling counterclockwise and it approaches the $\bar{y}\bar{z}$ plane. The spiral gets thinner on approach and then flattens out when reaching the $\bar{y}\bar{z}$ plane. Close to the $\bar{y}\bar{z}$ plane, the field line keeps on spiralling almost parallel to the $\bar{y}\bar{z}$ plane and then it runs off to the left, as in Figure 1.11. The large spiral in Figure 1.12 is almost the same as the spiral in Figure 1.11. The field line has to flatten out on approach of the $\bar{y}\bar{z}$ plane, since no field line can cross the symmetry plane. The field line pattern is reflection symmetric in the symmetry plane, so a mirror image spiral is present in the region $\bar{x} < 0$. This spiral has the same rotation direction, when viewed down the positive $\bar{x}$ axis. Very
close to the $\bar{y}\bar{z}$ plane, both spirals are almost identical to the big spiral in Figure 1.11. It is clear that neither field line comes out of the singularity at the center of the 2D vortex. Only for a field line exactly in the $\bar{y}\bar{z}$ plane does the vortex seem to have a source.

Figure 1.12  Shown is a 3D field line in the neighborhood of the large vortex in Figure 1.11. The axes origin has been moved for clarity of perspective.

1.10  Dipole off-center

A new aspect of the energy flow pattern appears when we consider a dipole located off the midway plane between the mirrors. Figure 1.13 shows field lines for $\delta = 4$, $h = 1$ and $\gamma = \pi / 3$. We see that all field lines come out of the dipole at the bottom-right, and some loop around the dipole, and enter the dipole at the other side. Just above the point of entry there is necessarily a singularity. The same phenomenon occurs for a single mirror [20], although under different conditions. A detailed analysis of this exotic effect will be presented elsewhere.
Figure 1.13  Shown are field lines for $\delta = 4$, $h = 1$ and $\gamma = \pi / 3$.

Tiny loops appear close to the dipole.

Figure 1.14 shows the field line pattern for $\delta = 4\pi$, $h = \pi / 2$ and $\gamma = \pi / 2$. For $\gamma = \pi / 2$ the pattern is reflection symmetric in the $\bar{z}$ axis, so only the region $\bar{y} \geq 0$ is shown. We notice the appearance of a large number of vortices. The field lines in the four vortices on the left rotate clockwise and the field lines in the vortices on the right rotate counterclockwise. This pattern repeats if we extend the graph to the right (not shown). Interestingly, if we replace $\delta = 4\pi$ by, say, $\delta = 12$, the entire pattern washes out, and we just get some wiggly curves going to the right. Similarly, if we would replace $\delta = 4$ in Figure 1.13 by $\delta = \pi$, a vortex appears to the left of the dipole. Apparently, the system is very sensitive to small variations in parameters.
Many vortices appear, more or less along vertical lines.

### 1.11 Locations of vortices and singularities

Energy flow patterns can become complicated, especially for a large separation $\delta$ between the mirrors, and details of the energy flow cannot always be resolved on the scale of the figure. For instance, in the flow pattern of Figure 1.14 there are 9 vortices and 11 singularities. At the center of a vortex is a singular point. At such a point the Poynting vector $\mathbf{\sigma}$ is necessarily zero. This can be due to $\mathbf{e} = 0$ or $\mathbf{b} = 0$ or $\mathbf{e}(\mathbf{r})^\dagger \times \mathbf{b}(\mathbf{r})$ imaginary. For field points in the $\bar{y}\bar{z}$ plane, vector $\mathbf{e}$ is in the $\bar{y}\bar{z}$ plane. Since $\mathbf{e}$ is the complex amplitude, the condition $\mathbf{e} = 0$ requires that the real and imaginary parts of both the $\bar{y}$ and the $\bar{z}$ components vanish simultaneously at a field point. Obviously, this is highly unlikely. The $\mathbf{b}$ vector, however, is along the $\bar{x}$ axis, so $\mathbf{b} = 0$ requires $\text{Re}[b_x] = 0$ and $\text{Im}[b_x] = 0$. Each equation defines a set of curves in the $\bar{y}\bar{z}$ plane, and at each
intersection we have a singular point. In Figure 1.15, the solid lines are the solution of $\text{Re}[b_x] = 0$ and the dashed lines are the solution of $\text{Im}[b_x] = 0$. The parameters are $\delta = 8$, $h = 4$ and $\gamma = \pi / 4$, which are the same as for the flow lines in Figure 1.10. The intersections of the curves, indicated by little black circles, appear to correspond to the locations of the four vortices in Figure 1.10. We have found in general that a singularity at the center of a vortex is due to the vanishing of the magnetic field at that point.

Besides the centers of vortices, numerous other singularities appear, and these are indicated by white circles in the figures. At such singularities, field lines split or collide, as can be seen for instance in Figures 1.4, 1.11 and 1.13. Here, we have $e(r)^* \times b(r)$ imaginary and this is the same as $\sigma = 0$. Since $\sigma$ is in the $\bar{y}\bar{z}$ plane, this is therefore the same as $\sigma_y = 0$ and $\sigma_z = 0$. Both equations define sets of curves, and at the intersections the Poynting vector vanishes. Figure 1.16 shows these curves for the same parameters as in Figure 1.15. Interestingly, the condition $\sigma = 0$ includes the condition $b = 0$, so the black circles in Figure 1.16 are at the same locations as the black circles in Figure 1.15. The sets of curves in both figures are very different, but they intersect at the same points for the locations of the vortices. In this fashion, we can tell which intersections in Figure 1.16 correspond to vortices and which intersections correspond to other singularities.

Near a mirror surface, the Poynting vector is parallel to the mirror and so $\sigma_z = 0$. Therefore, each mirror in the figure has a dashed line on top of it. Consequently, if a solid curve ends at a mirror, there is a singularity at this point. Two of these can be seen in Figure 1.16. As another example, Figure 1.17 shows the solution of $b = 0$ for $\delta = 4\pi$, $h = \pi / 2$ and $\gamma = \pi / 2$, and these are the same parameters as for the flow line picture of Figure 1.14. We see two sets of four vortices, more or less along upward lines. The one in the bottom-right corner is the lowest one of another set of four. Figure 1.18 shows the corresponding diagram for $\sigma = 0$. We notice 11 singularities, other than the
9 vortices. The locations of the vortices and singularities can very clearly be seen from Figure 1.18, whereas in the flow line picture of Figure 1.14, most of these locations are not clear at all.

Figure 1.15  The solid lines are the solution of $\text{Re}[b_x] = 0$ and the dashed lines are the solution of $\text{Im}[b_x] = 0$ and the parameters are the same as for the field line pattern in Figure 1.10.

The intersections, indicated by black circles, represent the locations of the four vortices.
The solid lines are the solution of $\sigma_y = 0$ and the dashes lines are the solution of $\sigma_z = 0$ and the parameters are the same as for Figure 1.15.

The black circles are the centers of vortices, and we conclude that by comparing to Figure 1.15. The white circles then must be singularities where field lines split.

The solid and dashed lines represent the solutions of $\text{Re}[b_x] = 0$ and $\text{Im}[b_x] = 0$, respectively, for $\delta = 4\pi$, $h = \pi / 2$ and $\gamma = \pi / 2$.

This corresponds to the flow line pattern of Figure 1.14.
Figure 1.18  The intersections shown in the figure are the singularities of the energy flow diagram of Figure 1.14.

1.12  Conclusions

The energy flow patterns for the propagation of electric dipole radiation in between parallel mirrors are far from trivial. The system only has three parameters: the dimensionless distance $\delta$ separating the mirrors, the dimensionless distance $h$ between the location of the dipole and the lower mirror and the angle $\gamma$ between the oscillation direction of the dipole moment and the normal to the mirror surfaces. For a radiating dipole in free space, most radiation is emitted perpendicular to the dipole axis (oscillation direction of the dipole moment) and none is emitted along the dipole axis. For a vertical dipole, as illustrated in Figure 1.3, this appears also to be the case for a radiating dipole in between mirrors, except that near the mirror surfaces the field lines of energy flow bend and become parallel to the surfaces. This has to be so, since no radiation can penetrate the mirrors. When we look further away from the dipole, however, as in Figures 1.4 and 1.5, singularities appear. First, two singularities appear near the midway line between the mirrors and, further out,
two singularities appear near the mirror surfaces. This pattern repeats indefinitely going outwards.
In the figures, we use dimensionless coordinates for which \(2\pi\) corresponds to an optical wavelength. Therefore, the patterns here are of near-wavelength or sub-wavelength scale.

For a horizontal dipole and small mirror separation we find again that most radiation is emitted perpendicular to the dipole axis and that the flow lines curve when they approach the mirrors, as shown in Figure 1.6. When the distance between the mirrors is increased, optical vortices appear in the vicinity of the dipole, as shown in Figure 1.8. When the dipole oscillates under a finite angle with the vertical, energy swirls around the dipole before taking off to infinity, as shown in Figure 1.9. For a larger separation between the mirrors we see from Figure 1.10 that several vortices appear in the flow pattern. Obviously, energy cannot accumulate at the center of a vortex and cannot come out of the singularity at the center of a vortex. It is shown in Figure 1.12 that the vortices in the plane of oscillation of the dipole should be seen as the cross sections of 3D vortices. Only in this plane is there a singularity at the center of the vortex. When the dipole is located off-center with the midpoint between the mirrors, small loops appear near the dipole. Field lines come out of the dipole at one side and then return back to the dipole at the other side. For larger separations between the mirrors, numerous vortices appear in the flow pattern of an off-center dipole.

When flow patterns become complicated, an alternative way to look at these patterns is by considering the locations of the vortices and singularities, without reference to the flow lines. At the singularity at the center of a vortex the magnetic field vanishes. This condition leads to two sets of curves and the vortices are located at the intersections. Typical examples are shown in Figures 1.15 and 1.17. Then, at any singularity, the Poynting vector is zero and this condition also leads to two sets of curves. Any singularity is located at intersections in these diagrams. Since the
singularities at the centers of the vortices are also reproduced, we have a means of distinguishing between vortices and points where the singularities are due to the splitting of field lines. Examples are shown in Figures 1.16 and 1.18, with the black circles indicating the locations of vortices and the white circles represent any other singularities.
1.13 References


CHAPTER II
DIPOLE RADIATION NEAR A REFLECTING CORNER

Adapted with permission from Henk F. Arnoldus, Zhangjin Xu and Xin Li, Dipole radiation near a reflecting corner, Journal of Applied Physics 127 (2020) 083101-1-12

We consider a radiating electric dipole, located near the joint of two orthogonal mirrors. It appears that the field lines of energy flow in the neighborhood of the dipole have an intriguing structure, depending on the state of oscillation of the dipole and its distance to each mirror. Numerous singularities and vortices appear in the sub-wavelength region between the dipole and the mirrors. We also present a method to find the locations of the vortices and singularities without regard to the details of the flow pattern. The radiation field induces a surface current density in the mirrors. It appears that the direction of the current is predominantly in the radial direction for a linear dipole, but it alternates between outgoing and incoming across singular curves. We show that the field line pattern expands with a phase velocity larger than the speed of light. For a circular dipole, there appears a spiral which runs inward. The current initially flows in along this spiral. Then the current leaves again along an outgoing spiral, which spirals inside the incoming spiral. Current can flow from one mirror to the other, and we show that the current always crosses the intersection line at a 90° angle.
2.1 Introduction

The rapid developments of nanotechnology and nano-photonics over the last few decades have made it paramount to study optical phenomena on a scale of a wavelength or even smaller. Ray diagrams cannot account for the interactions between nanoparticles and, for instance, a nearby substrate. Radiative emission rates of atoms, molecules and microparticles are affected by a nearby material medium, as was observed for the first time by Drexhage [1] in his landmark experiments with molecular dyes deposited on a dielectric material. The influence on molecular emission rates was studied theoretically in Ref. 2, and the predictions correspond very well with the experimental results. The recent advances in metamaterial design have opened an entirely new field of nanooptics. Negative-index-of-refraction materials have been predicted to possibly have the ability of focusing light tighter than the diffraction limit [3], and epsilon-near-zero metamaterials may provide a way to optically levitate a nanoparticle, provided it is located well within an optical wavelength from the interface with the material [4-6].

Of particular interest is the flow pattern of electromagnetic energy in a nano-size system. The simplest and most important source of electromagnetic radiation is the field emitted by an oscillating electric dipole. Already a free dipole can have an interesting radiation pattern in the optical near field. If the dipole moment rotates, as in a $\Delta m = \pm 1$ electronic transition in an atom, or as for a microparticle irradiated by a circularly-polarized laser beam, the energy emerges from the dipole as a vortex [7,8]. Within a wavelength from the dipole, the field lines of energy flow swirl around the axis through the center of the circle (perpendicular to the plane of rotation), and at larger distances the field lines become straight, like optical rays. This leads to a shift of the dipole image in the far field due to the rotation near the source [9]. This prediction has been confirmed experimentally [10].
The simplest substrate is a perfectly conducting mirror. The emission rate of a nearby molecule changes due to the presence of the substrate [11,12], and atomic levels shift due to the nearby metal [13,14]. Also of interest is the behavior of small particles in between two parallel mirrors. Level shifts and altered emission rates have been predicted [15-23], and suppression of spontaneous emission rates by atoms in between parallel mirrors has been confirmed experimentally [24-26]. The energy flow pattern of dipole radiation near a mirror has numerous vortices and singularities [27], and it was found that the vortices are arranged on four optical vortex strings [28]. When the dipole is placed in between parallel mirrors, intricate flow patterns are found [29], and it is predicted that radiation emerges from the dipole as a set of four vortices [30].

Figure 2.1  Shown is the setup of an electric dipole $\mathbf{d}(t)$ located near the junction of two perpendicular mirrors.

Vector $\mathbf{H}$ is the position vector of the dipole with respect to the origin of coordinates, and the dipole is located at the $yz$ plane. The mirrors have the $x$ axis in common.
Here we shall consider an electric dipole located near the junction of two mirrors, as depicted in Figure 2.1. The mirrors meet at a 90° angle, and the coordinate system is as shown. The dipole is located in the \(yz\) plane, and the mirrors are joint at the \(x\) axis. This setup was also considered in Ref. 31, where the spontaneous emission rate of an atom in this configuration was computed with a quantum electrodynamical approach.

### 2.2 Dipole radiation

An electric dipole, oscillating at angular frequency \(\omega\), has a dipole moment

\[
d(t) = d_o \text{Re} \left[ \hat{u} e^{-i\omega t} \right],
\]

where \(d_o > 0\), \(\hat{u}^* \cdot \hat{u} = 1\). If the unit vector \(\hat{u}\) is real, the oscillation is linear. For \(\hat{u}\) complex, the dipole moment vector \(d(t)\) will in general trace out an ellipse in a plane. For the complex amplitudes of the emitted and reflected electric and magnetic fields we shall split off factors as

\[
E(r) = \zeta e(r),
\]

\[
B(r) = \frac{\zeta}{c} b(r),
\]

with \(\zeta = k_o^3 d_o / (4\pi \varepsilon_o)\) and \(k_o = \omega / c\) is the wave number in free space. In this way, \(e(r)\) and \(b(r)\) are dimensionless, and overall constants are accounted for by the single parameter \(\zeta\). We also introduce dimensionless coordinates \(\bar{x} = k_o x, \bar{y} = k_o y\) and \(\bar{z} = k_o z\), and in terms of these
coordinates, a dimensionless distance of $2\pi$ represents an actual distance of an optical wavelength. The complex amplitudes of the electric and magnetic fields are [32]

$$
e(r) = \left\{ \hat{u} - (\hat{q} \cdot \hat{u})\hat{q} + [\hat{u} - 3(\hat{q} \cdot \hat{u})\hat{q}] \frac{i}{q} \left( 1 + \frac{i}{q} \right) \frac{e^{iq}}{q} \right\},$$

(2.4)

$$
b(r) = (\hat{q} \times \hat{u}) \left( 1 + \frac{i}{q} \right) \frac{e^{iq}}{q}.
$$

(2.5)

Here, $\mathbf{q} = k_o \mathbf{r}$ is the dimensionless position vector of the field point with respect to the location of the dipole, and $q = |\mathbf{q}|, \hat{q} = \mathbf{q} / q$.

\subsection*{2.3 The Image System}

For a dipole near a mirror, some of the emitted radiation reflects at the mirror, and the total field is the sum of the dipole field and the reflected field. The reflected electric and magnetic fields are identical to the fields produced by an image dipole below the mirror. This dipole is located at the same distance below the mirror as the dipole is above the mirror. The dipole moment of the image has its parallel component reversed, as compared to the dipole moment of the source dipole. So if we write $\mathbf{u} = \mathbf{u}_\perp + \mathbf{u}_\parallel$ for the dipole moment polarization vector of the source, then $\mathbf{u}^{\text{im}} = \mathbf{u}_\perp - \mathbf{u}_\parallel$ is the dipole moment polarization vector of the image. The situation is more complicated for the setup with the two mirrors shown in Figure 2.1. Let us write $\hat{\mathbf{u}} = (\hat{u}_x, \hat{u}_y, \hat{u}_z)$ for the Cartesian components of the source dipole unit vector, and indicate this as $\hat{\mathbf{u}}_1$. Let the image in the $xz$ plane be $\hat{\mathbf{u}}_2$. Reversing the parallel component of $\hat{\mathbf{u}}_1$ then gives $\hat{\mathbf{u}}_2 = (-\hat{u}_x, \hat{u}_y, -\hat{u}_z)$. But there is also an image in the $xy$ plane: $\hat{\mathbf{u}}_4 = (-\hat{u}_x, -\hat{u}_y, \hat{u}_z)$. This image produces a field in the vertical
mirror, which needs to be compensated for by the image of $\hat{u}_4$ in the vertical mirror. This is $\hat{u}_3 = (\hat{u}_x, -\hat{u}_y, -\hat{u}_z)$. The image system is summarized in Figure 1.2. Another way of looking at this is that the two lower dipoles are the images of the two top dipoles, and the two dipoles on the left are the images of the two dipoles on the right. The electric and magnetic fields of the four dipoles are given by Equation (2.4) and (2.5), respectively, with the appropriate vector $\hat{u}$, and vector $q$ is taken as the position vector of the field point with respect to the particular dipole. When we set $h = k_0H = (0, h_y, h_z)$, we have

\begin{align*}
q_1 &= (x, y - h_y, z - h_z), \\
q_2 &= (x, y + h_y, z - h_z), \\
q_3 &= (x, y + h_y, z + h_z), \\
q_4 &= (x, y - h_y, z + h_z),
\end{align*}

(2.6) (2.7) (2.8) (2.9)

as can be seen from the figure. The total electric and magnetic fields are then the sums of the four individual electric and magnetic fields.
2.4 Field lines of energy flow

Electromagnetic energy flows along the field lines of the Poynting vector

$$S(r) = \frac{1}{2\mu_o} \text{Re}[E(r)^* \times B(r)] .$$  \hspace{1cm} (2.10)

We introduce the dimensionless Poynting vector $\mathbf{\sigma} = 2\mu_o c S/\xi^2$, so that

$$\mathbf{\sigma} = \text{Re}[\mathbf{e}(r)^* \times \mathbf{b}(r)] ,$$  \hspace{1cm} (2.11)
which is now considered a function of the dimensionless field point coordinates \((x,y,z)\), or equivalently, a function of \(q\). We parametrize a field line of the vector field \(\mathbf{\sigma}\) by \(q(u)\), with \(u\) a dummy variable. The \(x\) coordinate of a point on the field line through a given point \((x_0,y_0,z_0)\) is then the solution of

\[
\frac{d}{du} \bar{x}(u) = \sigma_x[\bar{x}(u),\bar{y}(u),\bar{z}(u)],
\]

(2.12)

and similar equations hold for the \(y\) and \(z\) coordinates. This set of three equations can be solved numerically for a given initial point \((\bar{x}_0,\bar{y}_0,\bar{z}_0)\). The field lines run into the direction of increasing \(u\). Alternatively, field lines can be visualized with the Mathematica program StreamPlot. Near the dipole, the fields \(\mathbf{e}_1\) and \(\mathbf{b}_1\) (fields of the source dipole) diverge, which may give numerical problems. Field lines are only determined by the directions of the vectors of the vector field, so if we multiply \(\sigma\) by any positive function \(f\) of \(x\), \(y\) and \(z\), the field lines remain the same. A good choice seems to be \(f = q_1^5\).

Due to boundary conditions, near the surface of a mirror the electric field is perpendicular to the mirror surface, and the magnetic field is parallel to the surface. Therefore, near a mirror a field line of energy flow is parallel to the mirror surface. This should be so because energy cannot penetrate a perfectly conducting material.

### 2.5 Dipole and field lines in the \(yz\) plane

In this section, we shall consider a dipole oscillating or rotating in the \(yz\) plane, so that \(\hat{u}_x = 0\). It can then be checked from Equations (2.4) and (2.5) that for a field point in the \(yz\) plane
the electric field is in the $yz$ plane and the magnetic field only has an $x$ component. The corresponding Poynting vector at such a point is then in the $yz$ plane. Therefore, if a field line goes through a point in the $yz$ plane, it stays in the $yz$ plane. The field lines are 2D curves, whereas in general they will be 3D curves. For a linear dipole we set

$$\hat{u} = (0, \sin \gamma, \cos \gamma),$$  \hspace{1cm} (2.13)

so this corresponds to a dipole oscillating back and forth along vector $\hat{u}$, which makes an angle $\gamma$ with the positive $z$ axis.

Figure 2.3  Shown is the energy flow pattern for a linear dipole, oscillating under 60° with the $z$ axis.
A typical energy flow pattern is shown in Figure 2.3, where $\gamma = \pi/3$, $h_y = 10$ and $h_z = 5$. The field lines start at the location of the dipole. Far away from the dipole and the mirrors, field lines become straight. Near the intersection corner of the mirrors, the field lines need to bend so that they become parallel to the mirror surfaces. This leads to an intricate flow pattern with singularities and optical vortices.

Figure 2.4 shows an enlargement of the flow pattern near the corner. The black little circle is the center of the vortex, and the two little white circles are singularities where field lines split. Figure 2.5 shows an enlargement of the flow lines in the neighborhood of the dipole. There are two vortices and three other singularities. Interestingly, there are field lines that seem to start at the center of the upper vortex and end at the center of the lower vortex. Clearly, energy cannot be created at the center of the top vortex, and then accumulate at the lower vortex. The divergence of the Poynting vector is zero, so the field has no sources and sinks outside the dipole. All energy comes out of the dipole, and eventually flows away to infinity. The explanation of this phenomenon is illustrated in Figure 2.6. Shown is a 3D field line near the top vortex in Figure 2.5. The dipole is just outside the picture, located at $(0,10,5)$. The incoming field line is well off the $yz$ plane. Near the center of the vortex in the $yz$ plane, it swirls around, and then it leaves. The outgoing branch stays very close to the $yz$ plane, although that cannot be seen directly from the graph.

In Figure 2.7, the dipole is symmetrically placed with respect to the mirrors, and it oscillates under an angle of $\pi/4$ with the positive $z$ axis. The $45^\circ$ line is a singular line, and it can be shown that this is due to the vanishing of the magnetic field. Field lines come out of the dipole, bend to the dipole axis, and then stop at the diagonal line. In this figure, the dipole is located at $h_y = h_z =$
1.5. If we increase the distance to the origin by taking $h_y = h_z = 4$, then the field lines start at the diagonal singular line in the region above the dipole, but still end on the singular line in the region between the dipole and the mirror. This is shown in Figure 2.8.

Figure 2.4 The figure shows an enlargement of the vortex in the lower-left corner in Figure 2.3.
Figure 2.5  The figure shows an enlargement near the dipole of the flow line pattern in Figure 2.3.

Figure 2.6  Shown is a 3D field line near the top vortex in Figure 2.5.
Figure 2.7  Shown is the energy flow diagram for a dipole oscillating under 45° with the $z$ axis, and symmetrically placed with respect to the mirrors.

If we increase the distance further to $h_y = h_z = 6$, two vortices appear in the region between the dipole and the mirrors. This is shown in Figure 2.8. When the dipole is close to the mirrors, some field lines that come out of the dipole return to the dipole at the other side. These closed loop field lines are shown in Figure 2.9.
Figure 2.8  The same dipole of Figure 2.7 is now further away from the origin of coordinates, and two vortices appear.
2.6 Locations of vortices and singularities

The energy flow field line pattern is mainly determined by the vortices and singularities in the field of the Poynting vector. At the center of a vortex or any other singularity, the Poynting vector vanishes, leaving the direction of $\sigma$ undetermined. At the center of a vortex, the magnetic field vanishes [27]. In the yz plane, the complex amplitude of the magnetic field is along the x axis, so for $\mathbf{b}$ to vanish, both the real part and the imaginary part of $b_x$ have to be zero at the same point in the yz plane. We solve $\text{Re}[b_x] = 0$ and $\text{Im}[b_x] = 0$ numerically. The solution of each equation can be represented by a set of curves in the yz plane. We use solid curves for the solutions of $\text{Re}[b_x] = 0$ and dashed curves for the solutions of $\text{Im}[b_x] = 0$. At the intersection of a solid curve and a dashed curve, the magnetic field is zero, and this indicates the center of a vortex.
Figure 2.10 shows the solutions of $\text{Re}[b_x] = 0$ and $\text{Im}[b_x] = 0$ for the field line pattern in Figure 2.5, so for $\gamma = \pi / 3$, $h_y = 10$ and $h_z = 5$. The two little black circles indeed correspond to the locations of the two vortices in Figure 2.5.

At any singularity, $\sigma = 0$. Since $\sigma$ is in the $yz$ plane and real, we set $\sigma_y = 0$ and $\sigma_z = 0$ to find all singularities. The solutions of $\sigma_y = 0$ are represented by sets of solid curves, and the solutions of $\sigma_z = 0$ are indicated by dashed curves. At any intersection between a solid curve and a dashed curve we must have a singularity. Figure 2.11 shows the singularities for the flow lines in Figure 2.5. In the middle of the graph are four singularities, and comparison with Figure 2.10 then tells us which singularities are centers of vortices (little black circles) and which are other singularities (little white circles). At these other singularities we have points where field lines split, and it can be shown that at these points $\mathbf{e}^* \times \mathbf{b}$ is imaginary. On the $z$ axis, the Poynting vector is necessarily along the $z$ axis, so the $z$ axis is a solid curve. Similarly, the $y$ axis is a dashed curve. In Figure 2.11, the solid curve coming down from the dipole hits the $y$ axis, so this curve ends at a singularity on the $y$ axis. We see from Figure 2.5 that this is indeed a point where field lines split at the mirror surface.
Figure 2.10  The intersections between a solid curve and a dashed curve are points where the magnetic field vanishes, and these are the singular points at the centers of vortices.

Figure 2.11  The intersections between a solid curve and a dashed curve are points where the Poynting vector vanishes, and these are the singular points of the flow line pattern.

Most of the vortices and singularities appear in the area between the dipole and the corner of the two mirrors. Such flow patterns can be very complicated, and field line graphs are sometimes
not able to show sufficient detail, unless enlargements are made, as in Figure 2.4. We consider again a dipole located at $h_y = 10, h_z = 5$, but now we have

$$\hat{\mathbf{u}} = \frac{1}{\sqrt{2}} (0,1,i).$$

(2.14)

This is a dipole moment, rotating counterclockwise in the $yz$ plane. Figure 2.12 shows the locations of the vortices, and Figure 2.13 shows all singularities. Clearly, without considering the locations of the vortices first with Figure 2.12, we could never have been able to determine the nature of the singularities in Figure 2.13

![Figure 2.12](image)

Figure 2.12 Shown are the locations of vortices for a rotating dipole moment.
Figure 2.13  Shown are the locations of singularities for a rotating dipole moment.

2.7  Field lines in the plane of the horizontal mirror

We now consider field lines of the Poynting vector in the $xy$ plane. This vector is parallel to the surface at any point in the $xy$ plane, so the field lines are 2D curves in the $xy$ plane for any state of oscillation $\mathbf{u}$ of the dipole moment. A typical example is shown in Figure 2.14. As compared to the view in Figure 2.1, we now have the joint $x$ axes to the right, so we look ‘over the edge, from the left’ in Figure 2.1. The dashed little white circle is the projection of the dipole onto the $xy$ plane. For Figure 2.14, we have $h_y = 3$, $h_z = 1$, and the dipole moment rotates counterclockwise in the $xy$ plane. The dark line at the bottom of the figure is the $x$ axis, so the second mirror is perpendicular to this, and upwards. It seems that field lines are coming out of the $x$ axis on the left, and end on the $x$ axis on the right. In Appendix A, explicit expressions for the electric and magnetic field amplitudes for a field point in the $xy$ plane are given. When we set $y =$
0, we have \( q_2 = q_1 \), and with Equation (A1) this gives \( \mathbf{e} = 0 \), and therefore \( \sigma = 0 \). The \( x \) axis is a singular line, and field lines can end or start there. The magnetic field on the \( x \) axis is

\[
\mathbf{b} = 4 \mathbf{e}_x (h_x \hat{u}_y - h_y \hat{u}_z) \left( 1 + \frac{i}{q_1} \right) e^{i q_1} \frac{q_1}{q_1^2},
\]

(2.15)

which is along the \( x \) axis. We can also see this as follows. At the surface of a perfect conductor, \( \mathbf{e} \) has to be perpendicular to the surface. The mirrors are joint at the \( x \) axis, so \( \mathbf{e} \) has to be perpendicular to both surfaces. This is only possible if \( \mathbf{e} \) is zero. Similarly, \( \mathbf{b} \) has to be parallel to the surface of a perfect conductor. At the junction, this is only possible if \( \mathbf{b} \) is along the \( x \) axis. From \( \mathbf{e} = 0 \) we have \( \sigma = 0 \), no matter what \( \mathbf{b} \) is.

Just like for field lines in the \( yz \) plane, we can here draw curves with \( \sigma_x = 0 \) (solid lines) and \( \sigma_y = 0 \) (dashed lines). An example is shown in Figure 2.15, where \( \gamma = \frac{\pi}{3} \), \( h_y = 5 \) and \( h_z = 3 \). Since the dipole oscillates in the \( yz \) plane, the figure is symmetric between left and right. Here we have the peculiar situation that on the top part of the \( \Omega \)-shaped figure the solid and dashed lines coincide. Therefore, this part is a singular curve, and it can be seen that the field lines change direction across this curve. On the lower part of the \( \Omega \)-shaped figure the solid and dashed lines separate, and the field lines cross the solid curves vertically (hard to see in the figure) and they cross the dashed curves horizontally.
Figure 2.14  The figure shows field lines of the Poynting vector for a dipole moment, rotating in the $xy$ plane.

Figure 2.15  The figure shows field lines of the Poynting vector in the $xy$ plane for a dipole oscillating in the $yz$ plane, under an angle of 60° with the $z$ axis.

Another interesting case is shown in Figure 2.16. Here we have $\gamma = \pi/2$, so the dipole oscillates horizontally in the $yz$ plane, parallel to the lower mirror. We have $h_y = h_z = 10$. It can be shown from Equation (A1) that $\mathbf{e}$ is along the $z$ axis, $\mathbf{b}$ is along the $x$ axis, and therefore the Poynting vector is along the $y$ axis everywhere. We have $\sigma_x = 0$ everywhere, and so the “solid
line” is the entire mirror surface. Consequently, all points on a dashed curve do not only have \( \sigma_y = 0 \), but also \( \sigma_x = 0 \). Therefore, the dashed curves are singular curves, and we see from the figure that the field lines change direction across these curves.

![Figure 2.16](image)

Figure 2.16  Shown are field lines of the Poynting vector in the \( xy \) plane for a dipole oscillating parallel to the horizontal mirror and perpendicular to the vertical mirror.

### 2.8 Current density in the mirror

The magnetic field induces a surface current density in the horizontal mirror according to

\[
\mathbf{i}(\mathbf{r}, t) = \frac{1}{\mu_0} \mathbf{e}_z \times \mathbf{B}(\mathbf{r}, t),
\]

(2.16)

with \( \mathbf{B}(\mathbf{r}, t) \) the magnetic field just above the surface. Unlike the time-averaged Poynting vector, the current density varies with time, and its time average is zero. We define the dimensionless current density \( \mathbf{I}(\mathbf{r}, t) = \mathbf{i}(\mathbf{r}, t)/i_0 \), with \( i_0 = \zeta/(\mu_0c) \). We then have
\[ I(\mathbf{r}, t) = \text{Re}[\mathbf{e}_z \times \mathbf{b}(\mathbf{r}) e^{-i\omega t}]. \] (2.17)

The field \( \mathbf{b} \) at the surface is given by Equation (A2), and we obtain immediately

\[
I(\mathbf{r}, t) = \frac{2}{q_1^2} \text{Re} \left[ (q_{1,\|} \hat{u}_z + h_z \hat{u}_\|) (1 + \frac{i}{q_1}) e^{i(q_1 - \bar{t})} \right] \\
- \frac{2}{q_2^2} \text{Re} \left[ (q_{1,\|} \hat{u}_z + h_z \hat{u}_\| + 2(h_y \hat{u}_z - h_z \hat{u}_y) \mathbf{e}_y) \left( 1 + \frac{i}{q_1} \right) e^{i(q_1 - \bar{t})} \right],
\] (2.18)

which is now a function of \( \bar{x}, \bar{y} \) and the dimensionless time variable \( \bar{t} = \omega t \). In the figures we shall draw solid lines for the solutions of \( I_x = 0 \) and dashed lines for the solutions of \( I_y = 0 \).

As a first example, we set \( \hat{u} = \mathbf{e}_x \). Equation (2.18) then simplifies to

\[
I(\mathbf{r}, t) = 2h_z \mathbf{e}_x \left\{ \frac{1}{q_1^2} \left[ \cos(q_1 - \bar{t}) - \frac{1}{q_1} \sin(q_1 - \bar{t}) \right] \\
- \frac{1}{q_2^2} \left[ \cos(q_2 - \bar{t}) - \frac{1}{q_2} \sin(q_2 - \bar{t}) \right] \right\}.
\] (2.19)

The current density is in the positive or negative \( x \) direction, and the field line diagram is shown in Figure 2.17 for \( t = 0 \). Also shown are the solid lines, representing \( I_x = 0 \). These are singular curves of the field line pattern, and we see that the current density changes direction across a singular curve. For a point on the \( x \) axis we have \( q_1 = q_2 \), and we see from Equation (2.19) that this gives \( I = 0 \). Therefore, the \( x \) axis is a singular line. When \( t \) increases, the current density vector \( I(\mathbf{r}, t) \) oscillates back and forth in the \( x \) direction at a given field point \( \mathbf{r} \). This means that the singular curves \( I_x = 0 \) in Figure 2.17 change shape when time passes. Field lines run from one curve to the next one, and the directions alternate across each curve. So, if we can find how the
curves move with time, then the field lines just stay between the moving curves as in Figure 2.17. We shall consider the curves at field points with $\bar{y} \gg 1$, so at least several wavelengths away from the vertical mirror.

![Figure 2.17](image)

Figure 2.17  The figure shows field lines of the current density in the horizontal mirror at $t = 0$ for a dipole oscillating along the $x$ direction. The dipole is located at (0,10,5).

We also assume $\bar{y} \gg h$, so the field point is at least several wavelengths away from the location of the dipole. This corresponds to the top part of the pattern in Figure 2.17. From Equation (2.6) we have

$$q_1 = \sqrt{\bar{\rho}^2 + h^2 - 2h\bar{y}},$$

(2.20)
with $\bar{\rho} = \sqrt{x^2 + y^2}$ as the dimensionless distance to the origin. Far away this becomes

$$q_1 \approx \bar{\rho} - h_y \frac{\bar{y}}{\bar{\rho}} ,$$

(2.21)

and in the same way we find

$$q_2 \approx \bar{\rho} + h_y \frac{\bar{y}}{\bar{\rho}} .$$

(2.22)

This gives for the current density

$$I(\mathbf{r}, t) \approx \frac{4h_x}{\bar{\rho}^2} e_x \sin \left( h_y \frac{\bar{y}}{\bar{\rho}} \right) \left[ \sin(\bar{\rho} - \bar{t}) + \frac{1}{\bar{\rho}} \cos(\bar{\rho} - \bar{t}) \right] .$$

(2.23)

For a singular curve, the term in square brackets vanishes. This term only depends on $\bar{\rho}$, for a given $\bar{t}$. If we set $\bar{\rho} = \bar{R}$ for a solution, then the corresponding singular curve is a semi-circle with dimensionless radius $\bar{R}$. So, these radii are solutions of

$$\cot(\bar{R} - \bar{t}) = -\bar{R} .$$

(2.24)

When we differentiate Equation (2.24) with respect to $\bar{t}$, we obtain

$$\frac{d\bar{R}}{d\bar{t}} = 1 + \frac{1}{\bar{R}^2} .$$

(2.25)

The left-hand side is the rate of increase of $\bar{R}$, so this is the dimensionless speed at which this $\bar{R}$ grows. This is the radial speed at which the corresponding circle expands. We can see this as a
current density wave on the $xy$ plane, so that this speed is the phase velocity of the expanding wave. Going back to real variables (rather than dimensionless variables) then gives for the phase velocity of the expanding waves

$$v_{ph} = \left(1 + \frac{1}{R^2}\right)c,$$  (2.26)

with $c$ the speed of light. Clearly, the phase velocity is larger than the speed of light, and for $R$ large it approaches the speed of light. This means that circles with a small radius move faster than circles far away. That this should be so can be understood from Equation (2.24). When we graph the left-hand side and the right-hand side as a function of $R$, then the solutions are at the intersections of the two graphs. For $R$ large, the intersections are as good as at the locations of the asymptotes of the cotangent, so the solutions are approximately $R = n\pi$. At smaller distances, the values of $R$ are somewhat smaller, and so the spacing between the solutions is bigger. Therefore, a wave front (singular curve) with small $R$ has to cover more distance in the same time than a wave front with large $R$, and therefore it must have a larger phase velocity. This also shows that the singular curves are approximately $\pi$ apart, and this is half an optical wavelength.

Equation (2.15) gives the complex amplitude of the magnetic field on the $x$ axis. With Equation (2.17) we then find for the current density on the $x$ axis

$$I(r, t) = e_y\frac{4}{q_1^2} \text{Re} \left[ (h_x \hat{u}_y - h_y \hat{u}_x) \left(1 + \frac{i}{q_1}\right) e^{iq_1} \right],$$  (2.27)

and we see that it is in the positive or negative $y$ direction. Therefore, the current density flows towards the intersection of the mirrors or away from it, and under $90^\circ$. The current density is only
zero on the $x$ axis if $\mathbf{u} = \mathbf{e}_x$, as in the example above. Figure 2.18 shows the field lines of the current density for $\gamma = 60^\circ$ at $t = 0$. When time progresses, the singular curves expand, similar to the case of the linear dipole along the $x$ axis. At some locations on the $x$ axis the current flows towards the $x$ axis and at other places it flows away from it. Due to the scale of the figure, it cannot be seen what happens in the neighborhood of $x = 0$. An enlargement is shown in Figure 2.19, and we see that in this region the current flows in the positive $y$ direction. Obviously, current cannot be created at the $x$ axis, and the only possibility is that it flows towards the $xy$ plane along the vertical mirror. For the current density in the vertical mirror, just above the $x$ axis, we find

$$I(r, t) = -\mathbf{e}_z \frac{4}{\eta_1^2} \text{Re}\left[ (h_x \hat{u}_y - h_y \hat{u}_x) \left( 1 + \frac{i}{\eta_1} \right) e^{i\eta_1} \right].$$  \hspace{1cm} (2.28)

Here, the term in square brackets is the same as in Equation (2.27), so when the current from Equation (2.27) flows away from the $x$ axis, then the current from Equation (2.28) flows towards the $x$ axis. The current flowing down the vertical mirror hits the $x$ axis under $90^\circ$. Then it makes a $90^\circ$ turn and continues flowing away in the $xy$ plane, initially under $90^\circ$. Figure 2.20 shows the current density in the $xz$ plane for the same parameters as in Figures 2.18 and 2.19. The view here is that $x$ goes to the right, and $z$ goes up, so we look at the vertical mirror from behind, which is from the left in Figure 2.1.
Figure 2.18  Shown are field lines of the current density in the horizontal mirror for $\gamma = \pi / 3$ and $t = 0$. The dipole is located at (0,9,7).

Figure 2.19  The figure shows an enlargement of a part of the graph in Figure 2.18.
Figure 2.20  Shown are field lines of the current density in the vertical mirror.

At the bottom part of the picture the current flows towards the $x$ axis. There it makes a 90° turn towards the positive $y$ direction, and it continues flowing in the $xy$ plane, as shown in Figure 2.19.

Finally, we consider a dipole moment rotating in the $xy$ plane. For such a dipole, the polarization vector is $\hat{u} = (1, i, 0)/\sqrt{2}$. The current density in the $xy$ plane is shown in Figure 2.21, where we took the dipole to be located at $(0,10,5)$. There are no singularities or vortices for this case. Close inspection of the graph shows that the circular-looking curves are not closed loops. They actually form an infinite spiral. Starting from in on the right-hand side of the picture, the spiral is incoming, and spirals counterclockwise. At in on the left on the bottom, the spiral runs outside the picture, and returns near $\bar{x} = 5$ on the right. The curve spirals in until it passes through the projection of the dipole (dashed circle), and then it reverses direction to a clockwise rotation. At out near $\bar{x} = 10$ it leaves the picture, and then it continues inside the view to out on the left. Finally, it leaves the picture at out on the right. The clockwise outgoing spiral runs in between the
counterclockwise incoming spiral, and as such the whole picture is one curve, except that near the vertical mirror on the $x$ axis we imagine that it ‘continues’ outside the horizontal mirror. All field lines of the current density come in approximately over the incoming spiral. At some point along the incoming spiral, a field line makes turn, either left or right, and catches up with the outgoing spiral. Then the field line continues spiraling outward along this outgoing spiral. An enlargement of this splitting along the spiral in left and right going field lines is shown in Figure 2.22, and Figure 2.23 shows an enlargement of the transition from incoming to outgoing at the location of the dipole.

Figure 2.21 Field lines of the current density in the horizontal mirror for a dipole moment rotating in the $xy$ plane.
Figure 2.22  The figure shows an enlargement of a part of the flow line pattern of Figure 2.21.  The dipole is just in the top-left corner.

Figure 2.23  Shown is an enlargement of a part of Figure 2.21, in the neighborhood of the dipole.  The field lines going down in the top-left of the figure form the end of the incoming spiral.  At the dipole, they continue as the beginning of the outgoing spiral, running to the left under the dipole in the lower part of the diagram.
2.9 Conclusions

We have considered an oscillating electric dipole near the corner of two orthogonal mirrors, as depicted in Figure 2.1. The electric and magnetic fields in the space between the mirrors are identical to the fields by the dipole and three mirror images, as shown in Figure 2.2. We have studied the field lines of energy flow in the $yz$ plane for a dipole moment oscillating or rotating in the $yz$ plane. The field lines have very intricate structures, depending on the state of oscillation of the dipole and the distances to each of the mirrors. Numerous vortices and singularities appear. It was shown that field lines that seem to end at the center of a vortex are part of a 3D bundle of field lines that approach the dipole, and then swirl away after passing near the center of the vortex. The flow line patterns are largely determined by the presence of singularities and vortices in the flow field. We have shown that the locations of vortices can be found by solving for the points where the magnetic field complex amplitude vanishes. On solid curves in the figures, the real part of the magnetic field amplitude vanishes, and on dashed curves the imaginary part is zero. Any intersection of a solid curve with a dashed curve then signifies the location of a vortex. Similarly, by considering curves where the $y$ (solid curves) and $z$ (dashed curves) components of the Poynting vector are zero, any intersection of a solid curve and a dashed curve represents a singularity. By comparison with a graph showing where the magnetic field vanishes for the same parameters, we know which intersections indicate centers of vortices. The remaining singularities are points in the flow pattern where field lines split in different directions or where field lines end.

The dipole and its images induce a surface current density in the mirrors. Equation (2.18) gives an explicit expression for this current density in the horizontal mirror ($xy$ plane). We now draw curves in the current flow diagrams where the $x$ component (solid curves) or the $y$ component (dashed curves) of the surface current density is zero. At an intersection between a solid curve and
a dashed curve we have a singularity on the flow diagram. We see in Figure 2.18 that the dashed and solid curves are as good as identical, except close to the vertical mirror. For a linear dipole, field lines run approximately radially outward or inward, and a field line changes direction every time it crosses a singular curve. It was shown that the singular curves expand with time, with a phase velocity greater than the speed of light. In between singular curves, the current density keeps its direction (inward or outward), so the entire picture expands rapidly, with new singular curves being produced at the location of the projection of the dipole onto the $xy$ plane. Current flowing to the intersection of the mirrors hits the intersection line under $90^\circ$, and it continues to flow in the vertical mirror.

For a dipole moment rotating in the $xy$ plane, the current density also changes direction across curves, but these are not singular curves, and they are not closed curves. Field lines come in along an in-spiraling curve, and then suddenly deviate to the left or the right. They catch up with another arm of the same spiral, and this part of the spiral runs outward, in between the curves of the incoming part of the spiral. The spiral expands in time with a phase velocity larger than the speed of light, and the current flow field line pattern expands with it.
2.10 References


CHAPTER III

REFLECTION AND TRANSMISSION OF RADIATION BY AN EPSILON-NEAR-ZERO INTERFACE

Adapted with permission from Zhangjin Xu and Henk F. Arnoldus, Reflection by and transmission through an ENZ interface, OSA Continuum 2 (2019) 722-735

We have studied the reflection and transmission of traveling and evanescent plane waves, incident upon an ENZ material. The Fresnel reflection and transmission coefficients were obtained in the ENZ limit. For a $p$ polarized incident wave, the transmission coefficient vanishes, except very close to normal incidence. The reflection coefficient is $-1$ for both traveling and evanescent waves. It is shown, however, that there is a finite electric field in the ENZ material, even though the transmission coefficient is zero. This field is either linearly polarized or circularly polarized. The magnetic field in the medium for $p$ polarized illumination is zero, and therefore there can be no energy flow through the material. For $s$ polarization, the magnetic field in the medium is circularly polarized, and energy can flow through the material, parallel to the interface.

3.1 $\epsilon$-near-zero metamaterials

The electromagnetic properties of a linear isotropic homogeneous material are determined by the relative permittivity $\varepsilon$ and the relative permeability $\mu$. We shall suppress the common subscript $r$. These parameters of the medium are in general complex with a non-negative imaginary part. They depend on the angular frequency $\omega$ of the radiation under consideration. We shall
assume monochromatic irradiation, so that this frequency dependence becomes irrelevant. The index of refraction $n$ is a solution of

$$n^2 = \varepsilon \mu, \text{Im}[n] \geq 0. \quad (3.1)$$

This leaves an ambiguity when $\varepsilon$ and $\mu$ are both real and have the same sign. By including small positive imaginary parts in the parameters, we then find that $n$ should be taken as having the same sign as $\varepsilon$ and $\mu$ [1]. For natural occurring dielectrics, $\varepsilon$ and $\mu$ are positive, and so is $n$. Metamaterials are artificial media that can have, in principle, any values of $\varepsilon$ and $\mu$, with the restriction that their imaginary parts are non-negative. Ingenious sub-wavelength structures have been proposed theoretically and tested experimentally in order to construct materials with parameters $\varepsilon$ and $\mu$ that do not occur in nature.

In the past decades, a great deal of effort has been devoted to the development of so-called double negative metamaterials. For these materials $\varepsilon$ and $\mu$ are both negative (within evitable small positive imaginary parts), and therefore the index of refraction is approximately real and negative. These negative index of refraction materials (NIM’s) have been predicted to have peculiar properties by Vesalago [2]. A plane wave that refracts into the medium at an interface appears at the opposite side of the surface normal as compared to refraction into a regular dielectric with positive $n$. Later, Pendry [3] showed that incident evanescent waves could possibly be amplified by a slab of NIM, leaving a possible path to the construction of a superlens that can resolve images below the diffraction limit. Some controversies regarding the Fresnel reflection and transmission coefficients for such a layer were resolved in [4].
The Green’s function for the emission of radiation by a localized source, embedded in a medium with index of refraction $n$, is given by

$$g(r) = e^{ink_0r}/r,$$  
(3.2)

where $k_0 = \omega / c$ is the wave number in free space. Here, $r$ is the distance between the field point and a point inside the source. For a point source, like a dipole, located at the origin of coordinates, this is just the spherical coordinate $r$ of the field point. For a NIM with the real part of $n$ negative and the imaginary part of $n$ positive, this is an incoming spherical wave which damps out in amplitude in the outgoing direction. Obviously, the energy propagates outward, but the spherical wave carrying the energy travels inward. Similarly, the energy in a traveling plane wave propagates against the wave vector.

More recently, materials with epsilon-near-zero (ENZ) have attracted a great deal of attention. We shall assume that $\mu = 1$, so that $n = \sqrt{\varepsilon}$ is the correct solution of Equation (3.1). We then have $n \approx 0$, and we see from Equation (3.2) that the Green’s function for wave propagation in an ENZ material becomes $g(r) = 1/r$. The time dependence for a monochromatic field is taken as $\exp(-i\omega t)$. The expected spherical wave becomes $\exp(-i\omega t)/r$. The field oscillates with angular frequency $\omega$, but the spatial oscillations have disappeared. This phenomenon is referred to as ‘static optics’. The field still oscillates with time $t$, but the spatial part becomes $g(r) = 1/r$, which is the Green’s function of electrostatics. One may then wonder whether any energy transport is possible through such a material.

ENZ materials exist in nature. Most notably, if $\omega$ of the source is close to the plasma frequency of a metal, the $\varepsilon$ of the metal is close to zero. Below the plasma frequency the metal is
opaque and above the plasma frequency it is transparent. However, these plasma frequencies are in the UV (wavelength~130 nm for gold). Metamaterial ENZ media have been demonstrated in the microwave region \([5–7]\), for terahertz radiation \([8,9]\) and for the visible range of the spectrum \([10,11]\), and have been studied theoretically \([12–18]\). An interesting application of ENZ media is the possibility of squeezing, funneling or tunneling electromagnetic radiation through narrow channels or bends \([19–24]\). Also, waveform shaping and angular filtering has been proposed \([25-28]\), and levitation of small particles near the surface of an ENZ interface has been predicted \([29]\). Other applications include perfect optical absorbers \([30,31]\) and enhancement of the magneto-optical effect \([32]\).

The electric and magnetic fields emitted by a localized source can be represented in terms of the scalar Green’s function of Equation (3.2) \([33,34]\). When this function is represented by an angular spectrum \([35]\), the fields become superpositions (integrals) over plane waves. These waves are either traveling or evanescent, with respect to the surface of the material, and they are either \(s\) or \(p\) polarized. The reflected and transmitted plane waves can then be constructed with the help of appropriate Fresnel coefficients, and the reflected electric and magnetic fields then follow by superposition. This approach has proven to be very successful for the study of the radiation pattern of an electric dipole near an interface \([36]\), and for the computation of the field lines of energy flow in the near field \([37,38]\). In order to tackle this problem for an ENZ medium, it is imperative to study the solutions for plane waves first, and in particular include the evanescent incident plane waves.
3.2 Traveling and evanescent waves

We shall assume a harmonic time dependence with angular frequency $\omega$. Then the electric field is represented as

$$\mathbf{E}(\mathbf{r}, t) = \text{Re}[\mathbf{E}(\mathbf{r}) e^{-i\omega t}] ,$$

(3.3)

with $\mathbf{E}(\mathbf{r})$ the complex amplitude, and the magnetic field $\mathbf{B}(\mathbf{r}, t)$ is written similarly. The incident field is a plane wave with wave vector $\mathbf{k}_i$, as shown in Figure 3.1. It travels in vacuum towards an interface with a material having permittivity $\varepsilon$. We shall assume that the parallel component $k_\parallel$ of this wave vector is real, and given. A plane wave of an angular spectrum representation of a radiation field has this $k_\parallel$ as the free variable, and in the superposition this becomes the summation (integration) variable. Since the wave travels in vacuum, it has to hold that $\mathbf{k}_i \cdot \mathbf{k}_i = (\omega / c)^2$, and this is $\mathbf{k}_i \cdot \mathbf{k}_i = k_0^2$. When we set $\mathbf{k}_i = k_\parallel \mathbf{e}_z$ this gives $k_\parallel^2 + k_{i,z}^2 = k_0^2$, and this is an equation for $k_{i,z}$, since $k_\parallel = |\mathbf{k}_\parallel|$ and $k_0$ are given. We now introduce the parameter

$$\alpha = \frac{k_\parallel}{k_0} ,$$

(3.4)

and we set

$$\nu_1 = \sqrt{1 - \alpha^2} .$$

(3.5)

Then the two solutions for $k_{i,z}$ are $\pm k_0 \nu_1$. Parameter $\alpha$ is the dimensionless representation of the magnitude of $\mathbf{k}_\parallel$, and this is considered to be the free variable. For $0 < \alpha < 1$, we see that $\nu_1$ is
real and positive. Therefore, \( k_i \) is a real vector, and we have a traveling incident wave. We then take the solution \( k_{i,z} = k_0 v_1 \) for which the wave travels in the positive \( z \) direction. This wave is represented by arrow \( k_i \) in Figure 3.1. It is easy to see that \( \alpha = \sin \theta_i \), with \( \theta_i \) the angle of incidence. On the other hand, when \( \alpha > 1 \) we find that \( v_1 \) is positive imaginary. We take again the solution \( k_{i,z} = k_0 v_1 \), so that the wave decays exponentially in the positive \( z \) direction. The incident wave is now evanescent, and indicated schematically by the parallel lines in Figure 3.1. Such waves still travel along the surface with wave vector \( k_{||} \). The integration variable in an angular spectrum, after integration over the direction of \( k_{||} \) is the parameter \( \alpha \), with \( 0 < \alpha < \infty \).

![Figure 3.1 Illustration of a travelling or an evanescent plane wave reflecting off and transmitting into a dielectric, with the positive \( z \) direction as up.](image)

The wavevectors \( k_r \) and \( k_t \) of the reflected wave and the transmitted wave, respectively, must have the same parallel component \( k_{||} \) as the wavevector \( k_i \) of the incident wave. The wave vector of the incident wave is
\[ k_t = k_\parallel + k_o v_1 e_z , \]  
\[ (3.6) \]

and since the reflected wave is also in vacuum, we must have

\[ k_r = k_\parallel - k_o v_1 e_z . \]  
\[ (3.7) \]

For \( 0 < \alpha < 1 \) this wave is traveling and for \( \alpha > 1 \) the reflected wave is evanescent. The transmitted wave is in the medium with permittivity \( \varepsilon \), so we must have \( k_\parallel^2 + k^2_{t,z} = \varepsilon k_0^2 \). The causal solution is

\[ k_t = k_\parallel + k_o v_2 e_z , \]  
\[ (3.8) \]

with

\[ v_2 = \sqrt{n^2 - \alpha^2} , \]  
\[ (3.9) \]

and \( n = \sqrt{\varepsilon} \).

Let us now consider what kind of wave the \( t \) wave is. Let us assume for a moment that \( \varepsilon \) is positive, like in good approximation for most dielectrics. Then also \( n \) is positive. If \( \alpha < n \), then \( v_2 \) is real, and the \( t \) wave is traveling. If \( \alpha > n \), then \( v_2 \) is imaginary, and the \( t \) wave is evanescent.

If the incident wave is traveling, then \( \alpha \) is in the range \( 0 < \alpha < 1 \), and \( \alpha = \sin \theta_i \) with \( \theta_i \) the angle of incidence. If \( n > 1 \), then \( \alpha < n \) for all \( \alpha \), and thus the \( t \) wave is traveling. If \( n < 1 \), the \( t \) wave is traveling for \( 0 < \alpha < n \) and evanescent for \( n < \alpha < 1 \). Borderline is \( \alpha = n \), and this is
\[ \sin \theta_i = n . \] This is the familiar critical angle. If the incident wave is evanescent, we have \( \alpha > 1. \)

If \( n < 1, \) the \( t \) wave is evanescent for all \( \alpha. \) If \( n > 1, \) the \( t \) wave is evanescent for \( \alpha > n \) and traveling for \( 1 < \alpha < n. \) In the latter case we have the unusual situation that an evanescent incident wave is converted into a traveling wave upon transmission through the interface. For a metallic medium we have \( \varepsilon < 0, \) and \( n \) is positive imaginary. The \( t \) wave is evanescent for all \( \alpha. \) When \( \varepsilon \) is complex, with a positive imaginary part, then so is \( n. \) In addition, the real part of \( n \) is positive. Then the \( t \) wave is partially traveling and partially evanescent in the \( z \) direction.

### 3.3 Polarization vectors

When the incident wave is \( s \) polarized (TE), then so are the reflected and transmitted waves. The same holds for \( p \) polarization (TM). The unit vector in the \( \mathbf{k}_\parallel \) direction is

\[ \hat{\mathbf{k}}_\parallel = \frac{1}{\alpha k_o \mathbf{k}_\parallel} . \quad (3.10) \]

The unit vector for \( s \) polarization is taken to be

\[ \mathbf{e}_s = \mathbf{e}_z \times \hat{\mathbf{k}}_\parallel , \quad (3.11) \]

which is perpendicular to the plane of the page in Figure 3.1, and into the page. The unit vectors for \( p \) polarization are defined as

\[ \mathbf{e}_{p,l} = \frac{1}{k_o} \mathbf{k}_l \times \mathbf{e}_s , l = i, r , \quad (3.12) \]
\[ e_{p,t} = \frac{1}{n k_o} k_t \times e_s . \] (3.13)

With the wave vectors given by Equations (3.6–3.8), we find explicitly

\[ e_{p,i} = \alpha e_s - v_1 \hat{k}_\parallel , \] (3.14)

\[ e_{p,r} = \alpha e_s + v_1 \hat{k}_\parallel , \] (3.15)

\[ e_{p,t} = \frac{1}{n} (\alpha e_s - v_2 \hat{k}_\parallel) , \] (3.16)

and these lie in the plane of the paper in Figure 3.1, although they may be complex-valued. The normalization is chosen such that

\[ e_{\sigma,i} \cdot e_{\sigma,i} = 1 , \sigma = s,p , l = i,r,t . \] (3.17)

Furthermore, the polarization vectors are orthogonal to the corresponding wave vectors:

\[ e_{\sigma,i} \cdot k_l = 1 , \sigma = s,p , l = i,r,t , \] (3.18)

and obviously every s polarization vector is orthogonal to the corresponding p polarization vector.

### 3.4 Fields and Fresnel coefficients

The incident plane wave has wave vector \( k_i \) and amplitude \( E_o \), which may be complex. The expressions for the reflected and transmitted waves are plane waves with the corresponding
wave vectors and polarization vectors. The amplitudes are $E_0$ times the appropriate Fresnel coefficients. We find for $s$ polarization

$$
E(r)_1 = E_0 e^{ik_f \cdot r} e_s \left( e^{iv_1 z} + R_s e^{-iv_1 z} \right),
$$

(3.19)

$$
B(r)_1 = \frac{E_0}{c} e^{ik_f \cdot r} \left[ e_{p,t} e^{iv_1 z} + R_s e_{p,r} e^{-iv_1 z} \right],
$$

(3.20)

$$
E(r)_2 = E_0 e^{ik_f \cdot r} T_s e_s e^{iv_2 z},
$$

(3.21)

$$
B(r)_2 = n \frac{E_0}{c} e^{ik_f \cdot r} T_s e_{p,t} e^{iv_2 z},
$$

(3.22)

where the subscripts 1 and 2 refer to the vacuum and the medium, respectively. We have set $z = k_o z$ for the dimensionless $z$ coordinate of the field point. Similarly, for $p$ polarization we have

$$
E(r)_1 = E_0 e^{ik_f \cdot r} \left[ e_{p,t} e^{iv_1 z} + R_p e_{p,r} e^{-iv_1 z} \right],
$$

(3.23)

$$
B(r)_1 = -\frac{E_0}{c} e^{ik_f \cdot r} e_s \left( e^{iv_1 z} + R_p e^{-iv_1 z} \right),
$$

(3.24)

$$
E(r)_2 = E_0 e^{ik_f \cdot r} T_p e_{p,t} e^{iv_2 z},
$$

(3.25)

$$
B(r)_2 = -n \frac{E_0}{c} e^{ik_f \cdot r} T_p e_s e^{iv_2 z}.
$$

(3.26)

The magnetic field in the medium is proportional to $n$ for both polarizations. This may seem to lead to the conclusion that these fields vanish in the ENZ limit. We shall show below that this is
not necessarily the case. The expressions for the Fresnel coefficients follow by applying the usual boundary conditions at the interface. This yields

\[
R_s(\alpha) = \frac{v_1 - v_2}{v_1 + v_2},
\]
\[\text{(3.27)}\]

\[
T_s(\alpha) = \frac{2v_1}{v_1 + v_2},
\]
\[\text{(3.28)}\]

\[
R_p(\alpha) = \frac{\varepsilon v_1 - v_2}{\varepsilon v_1 + v_2},
\]
\[\text{(3.29)}\]

\[
T_p(\alpha) = \frac{2n v_1}{\varepsilon v_1 + v_2}.
\]
\[\text{(3.30)}\]

### 3.5 ENZ limit of the Fresnel coefficients

For an ENZ medium we have \(\varepsilon \to 0\) and \(n \to 0\). This gives \(v_2 \to i\alpha\) and \(v_1\) remains as is, Equation. (3.5). In Equations (3.27)–(3.29) we can immediately take this limit, with result

\[
R_s(\alpha)^{ENZ} = (v_1 - i\alpha)^2,
\]
\[\text{(3.31)}\]

\[
T_s(\alpha)^{ENZ} = 2v_1(v_1 - i\alpha),
\]
\[\text{(3.32)}\]

\[
R_p(\alpha)^{ENZ} = -1.
\]
\[\text{(3.33)}\]

The expression for \(T_p(\alpha)\) seems to go to zero for \(n \to 0\). However, if \(\alpha = 0\) (normal incidence of a traveling wave), then both numerator and denominator go to zero. Then \(v_1 = 1\) and \(v_2 = n\). With \(\varepsilon = n^2\) we then get \(T_p(\alpha)^{ENZ} = 2\). So
\[ T_p(\alpha)^{ENZ} = \begin{cases} 0, & \alpha \neq 0 \\ 2, & \alpha = 0 \end{cases} \] (3.34)

Let us now consider some of the properties of the Fresnel coefficients. In the following graphs, we plot the Fresnel coefficients as a function of \( \alpha \). The solid lines are the ENZ limits and the dashed lines are the values for a hypothetical medium with \( \varepsilon = 0.015 \times (1 + i) \). Figure 3.2 shows the real and imaginary parts of \( R_s(\alpha) \), and \( |R_s(\alpha)| \). For evanescent waves, the imaginary part in the ENZ limit is identically zero. We see that

\[ |R_s(\alpha)^{ENZ}| = 1, 0 \leq \alpha < 1, \]

as can also be shown explicitly with Equation (3.31). This means that a traveling wave undergoes a phase shift upon reflection, but the amplitude of the reflected wave is the same as the amplitude of the incident wave. For evanescent waves, however, the amplitude of the reflected wave is smaller than the amplitude of the incident wave, and the amplitude ratio decreases with \( \alpha \). Figure 3.3 shows the real and imaginary parts of \( T_s \), and \( |T_s| \). Again, the imaginary part is identically zero for evanescent waves. The real part, however, grows with \( \alpha \). For normal incidence we have \( T_s(0)^{ENZ} = 2 \), and for grazing incidence (\( \alpha = 1 \)), we have \( T_s(1)^{ENZ} = 0 \). For \( 0 \leq \alpha < 1 \), the incident wave is traveling, and \( v_1 \) is real, with \( 0 \leq v_1 < 1 \). Since \( v_1^2 = 1 - \alpha^2 \) we have \( |v_1 + i\alpha| = 1 \). Therefore, \( v_1 + i\alpha \) is on the unit circle in the complex plane. For the phase angle \( \theta \) we have \( \sin \theta = \alpha \). For a traveling wave, \( \alpha \) is the sine of the angle of incidence, so we conclude that this phase angle is \( \theta_i \). We then have
\[ v_1 + i\alpha = e^{i\theta_i}. \]  (3.36)

The Fresnel reflection and transmission coefficients from Equations (3.31) and (3.32) become

\[ R_s(\alpha)^{\text{ENZ}} = e^{-2i\theta_i}, \]  (3.37)

\[ T_s(\alpha)^{\text{ENZ}} = 2v_1e^{-i\theta_i}. \]  (3.38)

Relation (3.35) now follows immediately from Equation (3.37).

Let us now consider \( p \) waves. We see from Equation (3.33) that the reflection coefficient is \(-1\) for all \( \alpha \). Any \( p \) wave reflects back with only a phase shift, and this phase shift is the same for all \( \alpha \). This behavior is illustrated in Figure 3.4. The transmission coefficient vanishes for all \( \alpha \), except \( \alpha = 0 \), as shown in Figure 3.5. It seems that no radiation penetrates the material for \( \alpha \neq 0 \). This would be the same as for a perfect conductor (mirror). We shall show below that for an ENZ material this conclusion is incorrect. The dashed curves for the small value \( \varepsilon = 0.015 \times (1 + i) \) in Figure 3.5 are not close to the ENZ limit for traveling waves, which is zero for all \( \alpha \), except \( \alpha = 0 \). \( T_p(\alpha)^{\text{ENZ}} \) in Equation (3.34) has a discontinuity at normal incidence. Such a point singularity is obviously unphysical. The interpretation of this phenomenon can be inferred from Figure 3.5.

For \( \varepsilon \to 0 \), the imaginary part vanishes, and the real part has a sharp peak near \( \alpha = 0 \), with a height of 2. From Equation (3.9) for \( v_2 \) we see that in the ENZ limit and \( \alpha \to 0 \) we have \( v_2 \to 0 \). However, the limits \( n \to 0 \) and \( \alpha \to 0 \) do not commute. For \( n = 0 \) and \( \alpha \to 0 \) we have \( T_p(\alpha) \to 0 \), but for \( \alpha = 0 \) and \( n \to 0 \) we have \( T_p(0) \to 2 \). Physically, neither \( n \) nor \( \alpha \) can be exactly zero. Whether \( T_p(\alpha) \to 0 \) for \( n = 0 \) or \( T_p(0) \to 2 \) depends on which one is smaller: \( |n| \) or \( \alpha \). As mentioned in
Sec. 2, the critical angle (of incidence) is at $n = \alpha = \sin \theta_i$ (for $n$ real). For any $n$, there will always be a small range $0 \leq \alpha < |n|$ for which $T_p(\alpha) \approx 2$, and the transmitted wave is (partially) traveling. Outside this range, the transmitted wave is mainly evanescent, and $T_p(\alpha) \approx 0$. For $\epsilon = 0.015 \times (1 + i)$, we have $n = 0.13 + 0.056 \times i$ and $|n| = 0.15$ as the upper limit on $\alpha$. This corresponds to an angle of incidence of about $8^\circ$.

Finally, we mention that for $\alpha \approx 0$ the distinction between $s$ and $p$ waves should disappear. We see from Figures 3.2 and 3.4 that $R_s \approx 1$ and $R_p \approx -1$. The difference in sign is due to the phase conventions for the polarization vectors. From Figures 3.3 and 3.5 we see that $T_s \approx 2$ and $T_p \approx 2$. 
Figure 3.2  The figure on the left shows the real and imaginary parts of the reflection coefficient for $s$ waves.

The dashed lines are for the small value of $\varepsilon = 0.015 \times (1 + i)$, and the solid lines are the ENZ limits, with the real part red and the imaginary part blue. The figure on the right shows the ENZ limit of the absolute value of $R_s$ (green curve), with the dashed line the result for the same small $\varepsilon$ as in the figure on the left.

Figure 3.3  Shown are the real and imaginary parts of $T_s$, and the absolute value of $T_s$, for small $\varepsilon$ (dashed curves). The solid curves are the ENZ limits.
Figure 3.4   Shown are the real and imaginary parts of $R_p$, and the absolute value of $R_p$, for small $\varepsilon$ (dashed curves). The solid curves are the ENZ limits.

Figure 3.5   Shown are the real and imaginary parts of $T_p$, and the absolute value of $T_p$, for small $\varepsilon$ (dashed curves). The solid curves are the ENZ limits. For evanescent waves, the values of the real and imaginary parts of $T_p$ are close to their ENZ limit of zero, with the real part larger than the imaginary part. For traveling waves, the convergence to the ENZ limit is much slower. The dot represents $T_p(0)_{\text{ENZ}} = 2$. For $\alpha \neq 0$, we have $T_p = 0$, which is the green line in the figure on the right. In the figure on the left, the red and blue lines are on top of each other, making it look purple.
3.6 ENZ limit of the fields

We now consider the fields from Equations (3.19)–(3.26) in the ENZ limit. Nothing simplifies for the incident and reflected waves, except that \( R_p \to -1 \) in Equations (3.23) and (3.24), and \( R_s \) in Equations (3.19) and (3.20) can be replaced by the right-hand side of Equation (3.31). In the medium, \( v_2 \to i\alpha \), and so \( \exp(i v_2 z) \to \exp(-\alpha z) \). All fields in the ENZ medium are evanescent, with \( \alpha = 0 \) borderline. Let us first look at the \( s \) waves. In Equation (3.22), we have the product \( ne_{p,t} \). From Equation (3.16), we see that \( e_{p,t} \propto 1/n \), so the factor \( n \) cancels. We then get \( ne_{p,t} = \alpha e_z - v_2 \mathbf{k}_\parallel \), and with \( v_2 \to i\alpha \) this yields \( ne_{p,t} \to \alpha \mathbf{\eta} \), with

\[
\eta = e_z - i\mathbf{k}_\parallel .
\]

(3.39)

The \( t \) wave for \( s \) polarization becomes

\[
\mathbf{E}_2(r) = 2E_o v_1 (v_1 - i\alpha) \exp(i\mathbf{k}_\parallel \cdot \mathbf{r}) e_s \exp(-k_0 \alpha z) ,
\]

(3.40)

\[
\mathbf{B}_2(r) = 2\alpha \frac{E_o}{c} v_1 (v_1 - i\alpha) \exp(i\mathbf{k}_\parallel \cdot \mathbf{r}) \mathbf{\eta} \exp(-k_0 \alpha z) ,
\]

(3.41)

where we have replaced \( T_s \) by the right-hand side of Equation (3.32). The electric field is still \( s \) polarized, but the polarization vector \( e_{p,t} \) of the magnetic field has become \( \mathbf{\eta} \). In order to see the significance of this, we note that the right-hand side of Equation (3.38) is the complex amplitude of the magnetic field

\[
\mathbf{B}_2(r,t) = \text{Re}[\mathbf{B}_2(r)e^{-i\omega t}] .
\]

(3.42)
For a fixed point $\mathbf{r}$ in the ENZ medium we set temporarily

$$2E_0 \frac{\alpha}{c} v_1 (v_1 - i\alpha) e^{i k_1 \mathbf{r}} e^{-\alpha z} = Ae^{i \phi}, \ A > 0 , \ \phi \ \text{real} .$$  \hfill (3.43)

Then

$$\mathbf{B}_2 (\mathbf{r}, t) = A \text{Re} \{ \eta \exp[-i(\omega t - \phi)] \} ,$$  \hfill (3.44)

and this is

$$\mathbf{B}_2 (\mathbf{r}, t) = A [\mathbf{e}_z \cos(\omega t - \phi) - \hat{k}_\parallel \sin(\omega t - \phi)] .$$  \hfill (3.45)

Vector $\mathbf{B}_2 (\mathbf{r}, t)$ has magnitude $A$, and as a function of time it rotates in the plane of incidence in the direction from $\hat{k}_\parallel$ to $\mathbf{e}_x$. This is counterclockwise in Figure 3.1. The magnetic field in the medium is circularly polarized, although not in the usual sense. Now let us set

$$2E_0 \frac{\alpha}{c} v_1 (v_1 - i\alpha) = Ce^{i \psi} , \ C > 0 , \ \psi \ \text{real} ,$$  \hfill (3.46)

and take the $y$ axis along $\hat{k}_\parallel$. Then we have

$$\mathbf{B}_2 (\mathbf{r}, t) = C \exp(-\alpha z) \text{Re} \{ (\mathbf{e}_z - i\mathbf{e}_y) \exp[i(k_\parallel y - \omega t + \psi)] \} .$$  \hfill (3.47)
We first notice that the field has no \( x \) dependence, so it is constant into the direction perpendicular to the page in Figure 3.1. The magnetic field is the same in every plane parallel to the incident plane. The \( z \) dependence comes in as \( \exp(-\alpha z) \), so the amplitude decays exponentially into the direction perpendicular to the interface (up in Figure 3.1). The \( y \) and \( t \) dependence enters as \( \exp[i(k_\parallel y - \omega t + \phi)] \). This represents a traveling wave in the \( y \) direction (to the right in Figure 3.1), with phase velocity

\[
\nu_{ph} = \frac{\omega}{k_\parallel} = \frac{c}{\alpha},
\]

and wavelength \( \lambda = 2\pi / k_\parallel \). The dimensionless wavelength is \( \tilde{\lambda} = k_\circ \lambda \). So

\[
\tilde{\lambda} = \frac{2\pi}{\alpha}.
\]

In these units, the free space wavelength is \( \tilde{\lambda} = 2\pi \). If the incident wave is traveling, we have \( 0 < \alpha < 1 \), and the phase velocity along the surface is greater than the speed of light. If the incident wave is evanescent, we have \( \alpha > 1 \), and the phase velocity is less than the speed of light. The wavelength is greater than or less than the free space wavelength, respectively. Figure 3.6 shows field lines of the magnetic field in the plane of incidence for a fixed \( t \), and \( \alpha = 0.5 \). In each plane parallel to this plane the picture is the same. As a function of time, the field line pattern moves to the right with \( \nu_{ph} = 2c \). The picture is periodic in the horizontal direction with wavelength \( \tilde{\lambda} = 4\pi \). At the vacuum side, the pattern is due to interference between the incident and the reflected wave. The field lines form closed loops with singularities at the centers. Field lines that cross the interface stretch into the medium, and return to the interface to complete the loop in the vacuum
side. The magnitude of the magnetic field in the medium decreases exponentially in the upward direction, but this cannot be seen in a field line picture (field lines are determined by the direction of a vector field, but not by its magnitude). The electric field is s polarized, so its field lines are straight lines, perpendicular to the plane of the page in Figure 3.6.

Figure 3.6   Shown are the field lines of the magnetic field in the plane of incidence, for s polarization.

Here, $2\pi$ correspond to an optical wavelength in free space. The fat line in the middle is the interface, and the ENZ medium is above it.
For \( p \) waves we get the factor \( T_p \mathbf{e}_{p,t} \) in Equation (3.25). From Equations (3.16) and (3.30), we see that again a factor of \( n \) cancels. We therefore have two cases, just as for \( T_p \) in Equation (3.34). We now find

\[
\mathbf{E}_2(\mathbf{r}) = -2E_o \times \left\{ i v_1 \mathbf{n} \exp(i \mathbf{k}_\parallel \cdot \mathbf{r}) \exp(-\alpha z), \quad 0 < \alpha \ll |n|, \right. \quad \left. \alpha \gg |n|, \right. \tag{3.50}
\]

\[
\mathbf{B}_2(\mathbf{r}) = 0. \tag{3.51}
\]

The magnetic field vanished for all \( \alpha \). Here we have the remarkable situation that even though the transmission coefficient is zero for \( \alpha \gg |n| \), there is still a finite electric field in the material. For \( \alpha \gg |n| \), the electric field is circular polarized, and for \( 0 \leq \alpha \ll |n| \), this field is linearly polarized. For \( \alpha \to 0 \) the distinction between \( s \) and \( p \) waves should disappear. Comparison with Equations (3.40) and (3.41) shows that this is indeed the case, provided that for \( p \) waves we take the limit as \( 0 \leq \alpha \ll |n| \) in Equation (3.50). Experimentally, the value of \( n \) is fixed for a given material, and it can never be exactly equal to zero. However, \( \alpha = \sin \theta_i \), and the angle of incidence can be varied from large values to as good as zero. In the neighborhood of \( \sin \theta_i = |n| \), the polarization of the electric field changes abruptly from circular to linear, and the field becomes identical to the field for \( s \) waves in this limit.

### 3.7 Energy flow

We now return to the question whether or not energy will flow through the ENZ medium.

To this end, we consider the time-averaged Poynting vector.
\[ S(r) = \frac{1}{2\mu_0} \text{Re}[E(r)^* \times B(r)] , \] (3.52)

containing the complex amplitudes of the electric and magnetic fields. We spill off a factor

\[ S(r) = S_o S(r)' , \] (3.53)

with

\[ S_o = \frac{|E_o|^2}{2\mu_0 c} . \] (3.54)

In this way, \( S(r)' \) is dimensionless, and \( S_o \) is an overall measure for the field strength. It is advantageous to consider traveling and evanescent incident waves separately.

(a) \textit{Traveling incident wave}

For \( 0 \leq \alpha < 1 \), the incident wave is traveling, and parameter \( \nu_1 \) from Equation (3.5) is real and positive. In the complex conjugate of \( B(r) \) we can then set \( \nu_1^* = \nu_1 \). For the reflection and transmission coefficients in the ENZ limit we use Equations (3.37) and (3.38). We then obtain for \( s \) waves

\[ S(r)_1' = 4\alpha k_\parallel [\cos(\nu_1 \bar{z} + \theta)]^2 , \] (3.55)

\[ S(r)_2' = 4\alpha k_\parallel (1 - \alpha^2) e^{-2\alpha \bar{z}} . \] (3.56)
In the ENZ material, energy flows along the interface in the $\hat{k}_\parallel$ direction, and the magnitude of the energy flow vector decreases exponentially away from the interface. At the vacuum side, energy also flows in the $\hat{k}_\parallel$ direction with an oscillating $z$ dependence. At the interface we have $z = 0$, and with $(\cos \theta_i)^2 = 1 - \alpha^2$, we see that the Poynting vector is continuous across the interface. Energy is transported by the ENZ medium, but not in a direction away from the interface. For $p$ waves we find

$$ S(r)_1' = 4\alpha \hat{k}_\parallel [\sin(v_1\bar{z})]^2, \quad (3.57) $$

$$ S(r)_2' = 0. \quad (3.58) $$

The Poynting vector at the vacuum side is similar as that for $s$ waves, but now we find that there is no energy flow at all in the material for $p$ polarization. This is a direct consequence of the fact that the magnetic field in the medium is zero, as shown in Equation (3.51).

(b) Evanescent incident wave

For an evanescent incident wave, $v_1$ is positive imaginary, and we have $v_1^* = -v_1$. We define

$$ u = \sqrt{\alpha^2 - 1}, \quad (3.59) $$

So that $v_1 = iu$ and $u > 0$. We now find for $s$ waves
\[ S(r)_1' = \alpha \hat{k}_\parallel [e^{-uz} + R^{ENZ}_s e^{uz}]^2 , \]  
\[ S(r)_2' = \alpha \hat{k}_\parallel [T^{ENZ}_s]^2 e^{-2\alpha z} . \]  

For evanescent waves we have,

\[ R^{ENZ}_s = -(u - \alpha)^2 , \]  
\[ T^{ENZ}_s = -2u(u - \alpha) , \]

and both are real. It is easy to check that

\[ [1 + R^{ENZ}_s]^2 = [T^{ENZ}_s]^2 , \]

from which it follows that the Poynting vector is continuous across the interface. There is energy transport in the \( \mathbf{k}_\parallel \) direction near the interface. For \( p \) waves, we obtained that

\[ S(r)_1' = 4\alpha \hat{k}_\parallel [\sinh(u\hat{z})]^2 , \]  
\[ S(r)_2' = 0 . \]

We find again that for \( p \) waves no energy propagates through the material.
3.8 Normal incidence

Of practical importance is the case of normal or near normal incidence. In this case, the vector $\mathbf{k}_\parallel$ goes to zero, and $\hat{\mathbf{k}}_\parallel$ becomes undefined. We then assume a direction for $s$ polarization chosen, and set

$$\hat{\mathbf{k}}_\parallel = \mathbf{e}_s \times \mathbf{e}_z ,$$

as follows from Equation (3.11). For normal incidence, we have $\alpha = 0$, and the expressions for the fields simplify considerably. The time dependent fields follow from the complex amplitudes as in Equation (3.42).

With $v_1 = 1, R_s = 1, \mathbf{e}_{p,l} = -\hat{\mathbf{k}}_\parallel, \mathbf{e}_{p,r} = \hat{\mathbf{k}}_\parallel$ and $T_s = 2$ we find for $s$ polarization

$$\mathbf{E}(\mathbf{r}, t)_1 = 2E_0 \mathbf{e}_s \cos(\bar{z}) \cos(\omega t) ,$$

(3.68)

$$\mathbf{B}(\mathbf{r}, t)_1 = -2 \frac{E_0}{c} \hat{\mathbf{k}}_\parallel \sin(\bar{z}) \sin(\omega t) ,$$

(3.69)

$$\mathbf{E}(\mathbf{r}, t)_2 = 2E_0 \mathbf{e}_s \cos(\omega t) ,$$

(3.70)

$$\mathbf{B}(\mathbf{r}, t)_2 = 0 .$$

(3.71)

The electromagnetic field in vacuum is a standing wave, resulting from interference between the incident and reflected wave. Because the electric field in the ENZ material has no $\mathbf{r}$ dependence, its value is the same throughout the material, and oscillates with angular frequency $\omega$, which is called ‘static optics’. The magnetic field is zero everywhere.
For \( p \) waves near normal incidence

\[
E(r, t)_1 = -2E_o \hat{k} \cos(z) \cos(\omega t) ,
\]

\[
B(r, t)_1 = -2 \frac{E_o}{c} e_s \sin(z) \sin(\omega t) ,
\]

\[
E(r, t)_2 = -2E_o \begin{cases} \hat{k} \cos(\omega t) , & 0 \leq \alpha \ll |n| , \\ \eta \sin(\omega t) , & |n| \ll \alpha , \end{cases}
\]

\[
B(r, t)_2 = 0 .
\]

The electric field in the ENZ material is again 'static’. It is linearly polarized if \( \alpha \) is much closer to zero than \(|n|\), and circularly polarized if \(|n|\) is much closer to zero than \(\alpha\). When \( \alpha \) is seen as a variable, like in an angular spectrum, then there must be an abrupt, but smooth transition between the two polarization states.

Finally, we consider the time averaged Poynting vector in the limit of normal incidence. By considering the complex amplitudes in this limit, we find that in vacuum the value of \( E(r)^* \times B(r) \) is imaginary, and therefore the Poynting vector vanishes. In the ENZ medium, the magnetic field is zero. Therefore, for normal incidence we have \( S(r) = 0 \) everywhere.

### 3.9 Conclusions

We have studied the reflection off and the transmission through an ENZ interface. The incident field is taken as a plane wave, and we consider both traveling and evanescent incoming waves. The only variable in the problem is parameter \( \alpha \). For a traveling incident wave, \( 0 \leq \alpha < 1 \), this is the sine of the angle of incidence, and for an evanescent incident wave (\( \alpha > 1 \)), \( 1/\alpha \) is a
dimensionless measure for the penetration depth. Moreover, $k_0\alpha$ is the magnitude of the parallel components of all wave vectors (incident, reflected and transmitted).

We have obtained the ENZ limit of the Fresnel reflection and transmission coefficients. We found that for $\rho$ polarization of the incident wave the reflection coefficient is $-1$ for all $\alpha$. The transmission coefficient vanishes for all $\alpha$, except for $\alpha \ll |n|$, with both $\alpha$ and $n$ going to zero. Then the reflection coefficient is 2. Close to normal incidence ($\alpha = 0$) there is an abrupt but smooth transition in the transmission coefficient for $\rho$ waves. There is no such behavior for $s$ polarized incident radiation. In that case, the absolute value of the reflection coefficient is unity for traveling incident waves and for $\alpha > 1$ it drops to zero rapidly with increasing $\alpha$.

We then considered the electric and magnetic fields at both sides of the interface. At the vacuum side we have the usual interference between the incident and the reflected waves. In the ENZ medium all fields are evanescent for $\alpha \neq 0$. For an $s$ polarized incident wave, the electric field in the ENZ medium is also $s$ polarized. Remarkably, however, the magnetic field is exactly circularly polarized, and moves with a phase velocity $c/\alpha$ of along the surface. For $\rho$ polarized incident radiation, the magnetic field in the medium vanishes identically. The electric field is linearly polarized for $0 \leq \alpha \ll |n|$ and circularly polarized for $\alpha \gg |n|$. For $\alpha \gg |n|$, the transmission coefficient is zero for all $\alpha$, but there is still an electric field. For normal incidence, the magnetic field in the medium is zero, both for $s$ and $\rho$ irradiation. The electric field is ‘static’, either linearly or circularly polarized. The Poynting vector in the medium for $\rho$ waves is identically zero, but for $s$ polarization there is a finite flow of energy along the surface.
3.10 References


CHAPTER IV
FORCE ON AN ELECTRIC DIPOLE NEAR AN EPSILON-NEAR-ZERO INTERFACE

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An electric dipole near a surface is subject to the electromagnetic force by its own reflected field. We have derived a closed-form expression for this force for the case of an epsilon-near-zero (ENZ) medium. This force is perpendicular to the surface, and it is repulsive. We show that this force is due to the evanescent reflected field. For a rotating dipole moment, there is also a lateral force, which, in the ENZ limit, is vanishing small. However, when a small amount of damping is present in the material, this force becomes comparable with the force component perpendicular to the surface.

4.1 Introduction

Epsilon-near-zero materials are impenetrable materials for radiation in the usual sense. When a plane electromagnetic wave is incident on an ENZ interface, it creates a wave in the medium that travels along the surface and decays exponentially in amplitude in the direction away from the surface. For close to normal incidence, however, the electric component of the field extends into the medium, but the corresponding magnetic field is zero. This electric field oscillates with the frequency of the incident wave, but it has no spatial dependence. This bizarre phenomenon
is referred to as “static optics.” There is no energy flow into the material associated with this penetration of the electric field (Poynting vector is zero). The polarization of this electric field is the same as the polarization of the incident field (assumed to be s or p). However, for p polarization, and just off normal incidence, the polarization of the electric field becomes circular [1].

When the relative permittivity $\varepsilon_r$ of the medium vanishes, so does the index of refraction $n$ because $n = \sqrt{\varepsilon_r}$. The Green’s function for wave propagation is $g(r) = \exp(ink_or)$, with $k_o = \omega/c$ the wavenumber in free space, and $\omega$ is the angular frequency of the radiation. For an ENZ medium, this becomes $g(r) = 1/r$, which is the Green’s function of electrostatics. An oscillating electric field may exist in the medium, but there is no spatial dependence or retardation. As a result, an electric field can be squeezed or funneled through such a material [2–7] without loss of phase information. Due to the sharp, almost discontinuous, behavior near normal incidence, such materials can be used for angular filtering of radiation [8–11]. Enhancement of the magneto-optical effect [12] is another prediction. Metamaterials are artificial, subwavelength structures, designed in such a way that they effectively (macroscopically) behave as continuous media. The goal is then to design structures that behave as continuous ENZ media. Early attempts were restricted to low-frequency radiation [13–17], but more recently ENZ materials have been fabricated for the optical region of the spectrum [18–20].

Properties of atoms, molecules, and nanoparticles are modified when they are close to the interface with a medium. Early experiments by Drexhage [21] showed that emission rates by molecules are affected by the presence of a dielectric interface, and their radiation patterns are drastically altered due to interference between directly emitted radiation and the reflected radiation by the interface. Another direct consequence of the medium is that the reflected light exerts a force
on the particle. For a common dielectric or metal, this is an attractive force, which tends to make
the particle stick to the surface. It has been predicted [22,23] that, near an ENZ interface, this force
may be repulsive, leading to possible levitation of the particle. It has been argued [24] that this is
due to the expulsion of the electromagnetic field by the material, in analogy to the Meissner effect
for superconductors. We shall consider this force, and derive an explicit expression for it, for any
state of oscillation of the dipole. It will be shown that the force is (almost) entirely due to reflected
evanescent waves. We shall also show that the force is due to the phase shift upon reflection by
the ENZ medium rather than due to expulsion of the radiation by the material.

4.2 Force on an electric dipole

An electric dipole moment, oscillating at angular frequency $\omega$, can be represented as

$$d(t) = \text{Re}\left[d e^{-i\omega t}\right]. \quad (4.1)$$

Here, $d$ is the complex amplitude of the dipole moment $d(t)$. This dipole moment can be induced
through laser irradiation, oscillating at angular frequency $\omega$. The magnitude of vector $d$ is
determined by the laser power. For linear laser polarization, vector $d$ is real, and $d(t)$ oscillates
linearly. For circular or elliptic polarization, vector $d$ is complex, and $d(t)$ traces out a circle or an
ellipse in a plane. For a dipole near an interface, as considered below, $d$ can also have contributions
due to the presence of reflected light. Here, we shall take $d$ as given. The oscillating dipole emits
electromagnetic radiation, and the electric field must therefore have the form

$$E(r, t) = \text{Re}\left[E(r) e^{-i\omega t}\right], \quad (4.2)$$
with \( \mathbf{E}(\mathbf{r}) \) the complex amplitude. The magnetic field \( \mathbf{B}(\mathbf{r}, t) \) oscillates similarly. When the dipole is located at \( \mathbf{r}_o \) and is subject to an electric and magnetic field, the time-averaged force on the dipole is given by [25]

\[
\mathbf{F} = \frac{1}{2} \text{Re}[(\mathbf{d}^* \cdot \nabla)\mathbf{E}(\mathbf{r}) + i\omega \mathbf{d}^* \times \mathbf{B}(\mathbf{r})].
\] (4.3)

The first term is the Coulomb force by the electric field and the second term is the Lorentz force by the magnetic field. For time-harmonic fields, the magnetic field follows from the electric field according to Faraday’s law:

\[
\mathbf{B}(\mathbf{r}) = -\frac{i}{\omega} \nabla \times \mathbf{E}(\mathbf{r}),
\] (4.4)

and this allows the elimination of the magnetic field from Equation 4.3. We then obtain

\[
\mathbf{F} = \frac{1}{2} \text{Re}[(\mathbf{d}^* \cdot \nabla)\mathbf{E}(\mathbf{r})]_{r=r_o},
\] (4.5)

which offers a great computational simplification, as compared with Equation (4.3).

4.3 Dipole near an interface

An electric dipole is located a distance \( H \) above an interface with a medium, as illustrated in Figure 4.1. We take the dipole to be on the \( z \) axis, and the interface is the \( xy \) plane. The medium shown is a half-infinite dielectric, although that is irrelevant at this stage.
The dipole emits radiation, and this radiation is reflected at the surface. A most convenient representation of this reflected radiation is by means of an angular spectrum. It can be shown that \[ (4.6) \]

\[
E_r(r) = \frac{i k_o d_o}{8 \pi^2 \varepsilon_o} \sum_{\sigma=s,p} \int d^2 k_\parallel \frac{e^{i h v_1}}{v_1} (\hat{u} \cdot e_{\sigma,i}) R_{\sigma} e_{\sigma,r} e^{i k_r \cdot r},
\]

which holds without approximation. The reflected waves can be considered as superpositions of polarized ($\sigma = s, p$) plane waves, and the integration runs over the $k_\parallel$ plane. The wave vector of the reflected wave is $k_r = k_\parallel - i k_o v_1 e_z$. The unit vector in the $k_\parallel$ direction is $\hat{k}_\parallel$, and for the magnitude of the vector we set $k_\parallel = a k_o$. We have introduced $h = k_o H$ as the dimensionless distance between the surface and the dipole. The unit polarization vectors for the incident ($i$) and reflected ($r$) fields are chosen as
\[ \mathbf{e}_{s,l} = \mathbf{e}_{s,r} = \mathbf{e}_z \times \hat{k}_\parallel, \quad (4.7) \]

\[ \mathbf{e}_{p,l} = \alpha \mathbf{e}_z - v_1 \hat{k}_\parallel, \quad (4.8) \]

\[ \mathbf{e}_{p,r} = \alpha \mathbf{e}_z + v_1 \hat{k}_\parallel. \quad (4.9) \]

Here,

\[ v_1 = \sqrt{1 - \alpha^2}, \quad (4.10) \]

which has the significance of the dimensionless z component of the wave vector of the incident wave. For \( 0 \leq \alpha < 1 \), \( v_1 \) is positive, and both the incident and the reflected waves are traveling waves. For \( 1 < \alpha < \infty \), \( v_1 \) is positive imaginary, and both the incident wave and the reflected waves are evanescent. They decay exponentially in the directions away from the surface. The \( R_s \) and \( R_p \) are the Fresnel reflection coefficients for s and p polarized waves, respectively. They depend on the variable \( k_\parallel \) through its magnitude parameter \( \alpha \), but they do not depend on the direction of \( k_\parallel \). Expressions for these coefficients can be obtained from the appropriate boundary conditions at the interface.

For a single interface with a dielectric, as in Figure 4.1, the Fresnel coefficients are

\[ R_s = \frac{v_1 - v_3}{v_1 + v_3}, \quad (4.11) \]

\[ R_p = \frac{\varepsilon_r v_1 - v_3}{\varepsilon_r v_1 + v_3}, \quad (4.12) \]
with

\[ v_3 = \sqrt{\varepsilon_r - \alpha^2}, \quad (4.13) \]

as the dimensionless \( z \) component of the wave vector of the transmitted wave.

### 4.4 Evaluation of the force

The force on the dipole is exerted by its own reflected field. Expression (4.6) for \( \mathbf{E}_r(\mathbf{r}) \) is substituted into Equation (4.5) for the force. The gradient brings down the reflected wave vector as \( \mathbf{i} \mathbf{k}_r \). We adopt polar coordinates \((k_\parallel, \Phi)\) in the \( k_\parallel \) plane. Then, we have \( \mathbf{k}_\parallel = ak_0(\mathbf{e}_x \cos \Phi + \mathbf{e}_y \sin \Phi) \), and the polarization vectors from Equations (4.7) to (4.9) can be expressed in terms of \( \Phi \). The integrations over \( \Phi \) in the numerous terms are elementary. For the complex amplitude of the dipole moment, we set \( \mathbf{d} = d_o \mathbf{u} \), with \( d_o > 0 \), and vector \( \mathbf{u} \) is normalized as \( \mathbf{u} \cdot \mathbf{u} = 1 \). After regrouping, the expression for the force near an interface can be written in the attractive form

\[
\mathbf{F} = f_o \left[ (\mathbf{u}_\perp^* \cdot \mathbf{u}_\perp) v_\perp(h) + (\mathbf{u}_\parallel^* \cdot \mathbf{u}_\parallel) v_\parallel(h) \right] \mathbf{e}_z + f_o v_x(h) \mathbf{Im}[ (\mathbf{u}^* \cdot \mathbf{e}_z) \mathbf{u}_\parallel] .
\quad (4.14)
\]

Here, the overall constant

\[
f_o = \frac{d_o^2 k_0^2}{8\pi\varepsilon_o} ,
\quad (4.15)
\]

is a measure for the strength of the force. The three functions of \( h \) appearing in Equation (4.14) are
\[ v_\perp(h) = \text{Re} \left[ \int_0^\infty da \, \alpha^3 e^{2i\hbar v_1} R_p(\alpha) \right], \]  
(4.16)

\[ v_\parallel(h) = \frac{1}{2} \text{Re} \left\{ \int_0^\infty da \, \alpha e^{2i\hbar v_1} \left[ R_s(\alpha) - (1 - \alpha^2)R_p(\alpha) \right] \right\}, \]  
(4.17)

\[ v_\times(h) = -\text{Im} \left[ \int_0^\infty da \, \alpha^3 e^{2i\hbar v_1} R_p(\alpha) \right]. \]  
(4.18)

Equation (4.14) holds for any state of oscillation of the dipole, and no properties of the Fresnel coefficients have been used (other than rotational symmetry around the surface normal). For a perpendicular dipole, we have \( \hat{u} = e_z \), and Equation (4.14) reduces to \( \vec{F} = f_0 v_\perp(h) e_z \). The force is along the surface normal, either up or down, depending on the sign of \( v_\perp(h) \). Similarly, for a parallel dipole, the force is \( \vec{F} = f_0 v_\parallel(h) e_z \). In general, a dipole moment will have a perpendicular and parallel component, and Equation (4.14) shows how these mix up. Interestingly, there also appears a cross term containing the function \( v_\times(h) \). For this term to contribute, the vector \( \hat{u} \) must have both a perpendicular and a parallel component, and it has to be complex. For instance, for

\[ \hat{u} = -\frac{1}{\sqrt{2}}(e_y + i e_z), \]  
(4.19)

the dipole moment is rotating in the \( yz \) plane, and such that the dipole moment \( \vec{d}(t) \) rotates counterclockwise when viewed down the positive \( x \) axis. For the setup in Figure 4.1, this is clockwise. Then, \( \text{Im}[(\hat{u}^* e_z)\hat{u}_] = -e_y / \sqrt{2} \). There is a lateral force along the surface, which is in the plane of rotation. The direction of the force depends on the sign of \( v_\times(h) \). Such a lateral force on rotating dipoles has been predicted recently [27].
4.5 Force near an ENZ interface

For an ENZ material we have $\varepsilon_r = 0$, and the Fresnel coefficients from Equations (4.11) and (4.12) simplify to

$$R_s = (\sqrt{1 - \alpha^2} - i\alpha)^2,$$  \hspace{1cm} (4.20)

$$R_p = -1.$$ \hspace{1cm} (4.21)

For this case, the functions $v_i(h)$ become universal functions of $h$ in the sense that there is no dependence on any other parameters left. For an ENZ material, the integrals in Equations (4.16)–(4.18) can be evaluated explicitly, as we shall now show.

In order to evaluate the integrals over $\alpha$, we split them into integrals over the range $0 \leq \alpha < 1$ and over the range $1 < \alpha < \infty$. For the first range, the corresponding reflected waves are traveling waves; for the second range, the reflected waves are evanescent. The functions split accordingly, as for instance $v_\perp(h) = v_\perp(h)^{tr} + v_\perp(h)^{ev}$. For $0 \leq \alpha < 1$, we make the substitution $t = \sqrt{1 - \alpha^2}$; for $1 < \alpha < \infty$, we set $t = \sqrt{\alpha^2 - 1}$. For $v_\perp(h)$, we obtain

$$v_\perp(h) = \frac{2}{\beta^2} \left[ \left( 1 - \frac{3}{\beta^2} \right) \cos \beta - \frac{3}{\beta} \sin \beta \right],$$ \hspace{1cm} (4.22)

where we have set $\beta = 2h$. For the cross term we obtain

$$v_\times(h) = -\frac{2}{\beta^2} \left[ \frac{3}{\beta} \cos \beta + \left( 1 + \frac{3}{\beta^2} \right) \sin \beta \right].$$ \hspace{1cm} (4.23)
The result for \( v_{\parallel}(h) \) is somewhat more complicated. We find

\[
v_{\parallel}(h) = \frac{1}{\beta^2} \left(4 - \frac{9}{\beta^2}\right) \cos \beta + \frac{1}{\beta^2} \left(1 - \frac{9}{\beta^2}\right) \sin \beta + \frac{1}{15} \beta + L(\beta) \\
+ \frac{\pi}{2\beta^2} [H_2(\beta) - \beta H_3(\beta)].
\] (4.24)

Here, we introduced

\[
L(\beta) = \int_0^\infty dt \, t^2 \sqrt{1 + t^2} e^{-\beta t},
\] (4.25)

and this function is shown in Figure 4.2 as a function of \( h \). The Struve functions are defined as [28]

Figure 4.2  Shown is the function \( L(\beta) \) (solid line) as a function of \( h \) and its approximation by two terms (dashed line).
Figure 4.3  The figure shows the three functions that determine the force on the dipole near an ENZ medium, as a function of $h$.

\[ H_n(\beta) = \left( \frac{\beta}{2} \right)^{n+1} \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k + \frac{3}{2})\Gamma(k + n + \frac{3}{2})} \left( \frac{\beta}{2} \right)^2. \]  

(4.26)

The functions $v_\perp(h)$, $v_\parallel(h)$ and $v_\times(h)$ are shown in Figure 4.3. These functions, together with the dipole orientation vector $\hat{u}$, determine the force on the dipole near the ENZ interface, apart from the overall constant $f_0$. The functions $v_\perp(h)$ and $v_\parallel(h)$ are seen to be negative, except for small wiggles at large distances. This means that the force is repelling and upward in Figure 4.1. Because gravity is down, this force by the reflected field can give rise to levitation of the particle, holding it in suspension at a certain distance above the interface, as predicted in [24]. This will be a stable equilibrium. When the particle would move up a little, the electromagnetic force gets weaker, and gravity will pull it back to equilibrium. If the particle would move down a little, the electromagnetic repulsive force increases and will push the particle back up to equilibrium.
Because \( h = 2\pi \) corresponds to an optical wavelength in free space, we see that the repulsive force only extends outside the material over a fraction of a wavelength. This limits the particles to be suspended to atoms, molecules, and subwavelength nanoparticles. If the particle is a dielectric sphere, with a dipole moment induced by a moderate 50mW CW laser, we can estimate \( f_0 \) to be about \( 10^{-18} \) N. The force by gravity on this particle is also about \( 10^{-18} \) N, and levitation can be expected if the particle is close enough to the surface. We also see from the figure that the cross term is as good as negligible at short distances.

### 4.6 Role of evanescent waves

The results shown in Equations (4.22)–(4.24) represent the exact solution for the force on a dipole near an ENZ material. We shall now consider the contributions from the evanescent waves to these functions. These parts come from the integration range 1 < \( \alpha < \infty \) in Equations (4.16)–(4.18). We find, without approximation, that the evanescent parts are

\[
\begin{align*}
\nu_{\perp}(h)^{ev} &= -\frac{6}{\beta^4} - \frac{1}{\beta^2}, \\
\nu_{\parallel}(h)^{ev} &= -\frac{9}{\beta^4} - \frac{1}{2\beta^2} + L(\beta), \\
\nu_{\times}(h)^{ev} &= 0.
\end{align*}
\]  

We see from Figure 4.4 that the curve for \( \nu_{\perp}(h)^{ev} \) is almost identical to the curve for \( \nu_{\perp}(h) \). Therefore, the corresponding force is as good as determined by evanescent waves only. A similar conclusion holds for \( \nu_{\parallel}(h) \), as illustrated in Figure 4.5.
The appearance of the function $L(\beta)$ on the right-hand side of Equation (4.28) contributes significantly to the evanescent part of $v_\parallel(h)^{ev}$. To see this, we make the substitution $u = \beta t$ in Equation (4.25). Then, we expand the square root in a binomial series and integrate term by term. We then find

$$L(\beta) = \frac{6}{\beta^4} + \frac{1}{2 \beta^2} + \cdots. \quad (4.30)$$

Figure 4.4 Figure shows $v_\perp(h)$ (solid line) and its contribution from evanescent waves (dashed line), as a function of $h$. 
Figure 4.5  Figure shows $v_\parallel(h)$ (solid line) and its contribution from evanescent waves (dashed line), as a function of $h$.

When we substitute this in the right-hand side of Equation (4.28), the terms with $\beta^{-2}$ cancel, and the terms with $\beta^{-4}$ combine as

$$v_\parallel(h)^{ev} = -\frac{3}{\beta^4} + \cdots .$$  \hspace{1cm} (4.31)

When graphing $-3\beta^{-4}$, the result is identical to the dashed curve in Figure 4.5. Interestingly, the evanescent waves do not contribute to the cross term, as seen in Equation (4.29).

4.7 Contributions from traveling waves

The traveling part of $v_\perp(h)$ is

$$v_{\perp}^{tr}(h) = v_\perp(h) - v_\perp^{ev}(h) ,$$ \hspace{1cm} (4.32)

and we find with Equations (4.22) and (4.27)
\[ v_{\perp}(h)^{tr} = \frac{6}{\beta^4} + \frac{2}{\beta^2} \left[ \left( 1 - \frac{3}{\beta^2} \right) \cos \beta - \frac{3}{\beta^2} \sin \beta + \frac{1}{2} \right]. \]  

(4.33)

It seems that there are many negative powers of \( \beta \), suggesting that \( v_{\perp}(h)^{tr} \) diverges for small \( \beta \), just as \( v_{\perp}(h)^{ev} \). However, when we expand \( \cos \beta \) and \( \sin \beta \) in series around \( \beta = 0 \) and keep a large number of terms, we find that all terms with negative powers cancel exactly, and we are left with

\[ v_{\perp}(h)^{tr} = -\frac{1}{4} + O(\beta). \]  

(4.34)

Apparently, the traveling contribution is finite for \( \beta \) small, and \( v_{\perp}(h) \) is determined by the diverging contribution from evanescent reflected waves.

In order to see that traveling contributions must be finite, we consider the representation

\[ v_{\perp}(h) = -\text{Re} \left[ \int_0^\infty d\alpha \, \alpha^3 e^{2ihv_1} \right], \]  

(4.35)

from Equation (4.16), and with \( R_p(\alpha) = -1 \) for an ENZ medium. The traveling part comes from the integration range \( 0 \leq \alpha < 1 \). Here, we make the substitution \( t = \sqrt{1 - \alpha^2} \). This yields the representation

\[ v_{\perp}(h)^{tr} = -\int_0^1 dt \, t(1 - t^2) \cos \beta t. \]  

(4.36)

Then, we expand \( \cos \beta t \) in a series and integrate term by term. We then find the representation
\[ v_\perp(h)^{tr} = -\frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{(2n)! (n+1)(n+2)} (-\beta^2)^n. \] (4.37)

The first few terms are

\[ v_\perp(h)^{tr} = -\frac{1}{4} + \frac{1}{24} \beta^2 - \frac{1}{576} \beta^4 + \cdots. \] (4.38)

Along the same lines, we find

\[ v_\parallel(h)^{tr} = \frac{1}{8} + \frac{2}{15} \beta - \frac{3}{48} \beta^2 - \frac{4}{315} \beta^3 + \cdots, \] (4.39)

\[ v_\times(h)^{tr} = \frac{2}{15} \beta - \frac{1}{105} \beta^3 + \cdots. \] (4.40)

For the cross term, the evanescent waves do not contribute, so the terms shown here are the dominant terms for \( v_\perp(h)^{tr} \) at small distances. For \( v_\parallel(h)^{tr} \) and \( v_\times(h)^{tr} \), the traveling contributions only produce slight wiggles in the curves for large distances.

### 4.8 Epsilon near zero

The above results hold without approximation for the force on a dipole near an ENZ medium. For such a medium, \( \varepsilon_r = 0 \) (and \( \mu_r = 1 \)). In practice, such a perfect ENZ material does not exist. The value of \( \text{Re}[\varepsilon_r] = 0 \) should be close to zero, and so should the value of \( \text{Im}[\varepsilon_r] = 0 \), in order to be considered ENZ material. We shall now consider the possible effects of small
deviations from the perfect ENZ limit. We use the Fresnel coefficients from Equations (4.11) and (4.12), and the integrations in Equations (4.16)–(4.18) are performed numerically.

First, we consider the result for $v_\perp(h)$. Figure 4.6 shows $v_\perp(h)$ in the ENZ limit (solid line), which is the same curve as in Figure 4.3. The dashed line is $v_\perp(h)$ for $\varepsilon_r = 0.5$. We see that the curve is somewhat shifted to the left, but the main features are the same. The corresponding force is still repulsive, but the magnitude of the force is slightly smaller for a given distance. When we set $\varepsilon_r = 0.5$, the curve shifts somewhat to the right, thereby increasing the force. When we set $\varepsilon_r = 0.5i$, representing damping in the material, there is hardly any change, as compared with the ENZ limit. The same conclusions hold for $v_\parallel(h)$.

Figure 4.6 Shown is $v_\perp(h)$ in the ENZ limit (solid line) and for $\varepsilon_r = 0.5$ (dashed curve).
Shown is $v_x(h)$ in the ENZ limit (solid line) and for $\varepsilon_r = 0.5$ (dashed curve).

For $v_x(h)$, the force vanishes when the dipole gets close to the surface, as shown in Figure 4.3. It also follows from Equation (4.40) with $\beta \rightarrow 0$. Figure 4.7 shows the effect of a small positive value of $\varepsilon_r$. The lateral force no longer goes to zero for $h \rightarrow 0$, and the $h$ dependence is significantly different than in the ENZ limit.

When we set $\varepsilon_r = -0.5$, the curve ends at a negative value for $h \rightarrow 0$. More interesting is the effect of damping, as is illustrated in Figure 4.8. Here, we took $\varepsilon_r = 0.01i$, and the change is dramatic. Even with this minuscule value of $\text{Im}[\varepsilon_r]$, the value of $v_x(h)$ diverges for $h \rightarrow 0$, just as it always does for $v_\perp(h)$ and $v_\parallel(h)$. The lateral force is $f_0 v_x(h) \text{Im}[((\hat{u}^* \cdot e_z) \hat{u}_\parallel)]$, so, given $\hat{u}$, its direction is determined by the sign of $v_x(h)$. When we take $\hat{u}$ as in Equation (4.19), we have $\text{Im}[(\hat{u}^* \cdot e_z) \hat{u}_\parallel] = -e_y / \sqrt{2}$, and because $v_x(h)$ is negative, the force is in the positive $y$ direction.

Referring to the view in Figure 4.1, the dipole moment rotates clockwise, and the lateral force is to the right.
It was shown above that $v_x(h)$ has only a contribution from traveling waves in the ENZ limit. For $\text{Im}[\varepsilon_r] > 0$, the evanescent waves kick in, such that the lateral force becomes large when the dipole is close to the interface. Practically, there will always be small damping in the material. Then, the lateral force becomes comparable with the perpendicular force, produced by the perpendicular and parallel parts of the dipole moment orientation vector $\mathbf{u}$. In addition, there is a lateral force by the laser beam, which illuminates the particle to induce the dipole moment. With $\mathbf{u}$ from Equation (4.19), this radiation must be circularly polarized and propagate along the $x$ axis, as shown in Figure 4.1. This radiation pressure force is in the propagation direction of the beam.

Figure 4.8  Shown is $v_x(h)$ in the ENZ limit (solid line) and for $\varepsilon_r = 0.01i$ (dashed curve).

4.9  Mirror

An ENZ material is as good as impenetrable for electromagnetic radiation. Only just below the surface, waves can travel along the surface as evanescent waves. No energy is transported into the material in the direction normal to the surface. It is tempting to speculate that this would lead
to levitation [24]. The emitted radiation by the dipole in the downward direction has nowhere to go and produces a cushion for the particle. We shall now show that the mechanism for levitation is more subtle.

The ultimate impenetrable material is a perfect conductor (mirror). Not even evanescent waves can move below the surface through the material. In this mirror limit, the Fresnel coefficients are $R_s = -1, R_p = 1$. There is no dependence on the integration variable $\alpha$, and the integrals in Equations (4.16)–(4.18) can be computed easily. We obtain

\begin{equation}
    v_\perp(h) = -\frac{2}{\beta^2} \left[ \left( 1 - \frac{3}{\beta^2} \right) \cos \beta - \frac{3}{\beta} \sin \beta \right],
\end{equation}

\begin{equation}
    v_\parallel(h) = -\frac{\sin \beta}{\beta} + \frac{1}{\beta^2} \left[ \left( \frac{3}{\beta^2} - 2 \right) \cos \beta + \frac{3}{\beta} \sin \beta \right],
\end{equation}

\begin{equation}
    v_\times(h) = \frac{2}{\beta^2} \left[ \frac{3}{\beta^2} \cos \beta + \left( 1 - \frac{3}{\beta^2} \right) \sin \beta \right],
\end{equation}

and Figure 4.9 shows the corresponding curves. There is a striking resemblance between the curves in Figure 4.9 for a mirror and the curves in Figure 4.3 for an ENZ material. The graphs are almost identical but inverted with respect to the $h$ axis. For $v_\perp(h)$ and $v_\times(h)$, this follows immediately from Equations (4.16) and (4.18). Only the Fresnel coefficient $R_p$ appears in the expressions, and they are of opposite sign for a mirror and an ENZ material. For $v_\parallel(h)$, there is a slight difference between the ENZ result and the mirror result, although this can barely be seen in the figures. Clearly, when we have levitation for an ENZ medium, we must have attraction for a perfect conductor. The forces are as good as identical but of opposite signs. This shows that the difference
between repulsion and attraction is determined by the phase shift upon reflection and not by expulsion of the radiation by the interface.

![Figure 4.9](image)

**Figure 4.9** The figure shows the functions $v_\perp(h)$, $v_\parallel(h)$ and $v_\times(h)$ for a perfect conductor, as a function of $h$.

### 4.10 Conclusions

An electric dipole near an interface experiences a force by its own reflected radiation. We have derived a closed-form expression for this force for the case where the medium is an ENZ material. The result holds for any state of oscillation or rotation of the dipole moment. The force is (mainly) perpendicular to the surface, and it is shown that for close (subwavelength) distances between the dipole and the interface, this force is repulsive. It is shown that the force is exerted on the dipole by the reflected evanescent waves of the angular spectrum of the radiation. The traveling waves only contribute minimally to some small wiggles for large distances. The force at close distances is proportional to $h^{-4}$, where $h$ is the dimensionless distance between the dipole and the surface (on such a scale, $2\pi$ corresponds to one optical wavelength). A cross term between the perpendicular and parallel components of the dipole moment appears in the expression for the force. For this term to contribute, the dipole moment needs to rotate (circle or ellipse) and have
both a perpendicular and parallel component with respect to the surface. The resulting force is lateral (parallel to the surface), no matter the state or plane of rotation of the dipole moment. In the ENZ limit, this force only comes from the traveling waves in the angular spectrum and is, consequently, very small compared with the force perpendicular to the surface. However, when the slightest amount of damping in the material is present, this lateral force acquires a contribution from evanescent waves and becomes comparable with the normal force.
4.11 References


CHAPTER V

ELECTRIC DIPOLE POWER EMISSION NEAR AN ENZ MEDIUM

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Power emission by an electric dipole near an interface is considered. We derive explicit expressions for the emitted power for any state of oscillation of the dipole, without making use of the material properties of the substrate, and we derive an expression for the power crossing the interface. It is shown that the power naturally splits in contributions from traveling and evanescent incident waves. We then consider an ENZ material and obtain explicit expressions for the power. It is shown that only traveling waves contribute, and that no power crosses into the material. When the slightest amount of absorption is present in the medium, the evanescent waves kick in, and in such a way that the emitted power diverges when the distance between the dipole and the particle becomes small.

5.1 Introduction

It was realized for the first time by Purcell [1], in 1946, that the energy emission rate of an atom, during spontaneous decay, is not only determined by the intrinsic properties of the atom, but also by its environment. In quantum electrodynamics, spontaneous emission of a photon, together with spontaneous decay of the atom from an excited electronic state to the ground state, is brought about by the coupling of the electric dipole moment of the atom to the local electric field. Both the
dipole moment and the electric field are operators, and this leads to an interaction Hamiltonian between the atom and the field. Solution of the Schrödinger equation, either by perturbation theory or reservoir theory, then leads to a prediction of the energy emission rate. When the atom is located near an interface or confined to a cavity, the vacuum is altered, and this will affect the emission rate. In particular, the distance between the atom and the surface will be an important factor, and also the electronic properties of the medium will come into play. Such alterations of the emission rate have been observed experimentally for Rydberg atoms in a cavity [2] and in between parallel mirrors [3]. Similarly, the fluorescence emission rate of molecules near a substrate is affected by the material, as was experimentally confirmed in the celebrated experiments by Drexhage [4]. Numerous theoretical approaches have been presented [5,6], with the angular spectrum approach [7,8] being the most powerful and general. Except for subtle effects on the atomic scale, a classical approach is adequate for the power emission problem.

For atoms, molecules, nano- and microparticles, the electric dipole moment will provide the dominant mode of radiation. When such a small particle is irradiated by a laser beam of angular frequency $\omega$, an electric dipole moment will be induced. With $d$ the complex amplitude of the oscillating dipole moment, the emitted power is given by [9]

$$P_o = \frac{\omega^4}{12\pi \varepsilon_0 c^3} d^* \cdot d,$$

(5.1)

when the particle is surrounded by free space only. We shall set $d = d_o \hat{u}$, with $d_o > 0$ and $\hat{u}^* \cdot \hat{u} = 1$. Then the amplitude $d_o$ is determined by the laser power, and the polarization $\hat{u}$ of the dipole moment is determined by the polarization of the laser electric field. When $\hat{u}$ is real, apart from a
possible overall phase factor, the oscillation is linear. When \( \hat{\mathbf{u}} \) is complex, the dipole moment rotates, and traces out an ellipse in a plane.

The time-averaged emitted power by an oscillating electric dipole can be given by [10]

\[
P = \frac{1}{2} \omega \text{Im}[\mathbf{d}^* \cdot \mathbf{E}(\mathbf{r}_o)]
\]  

(5.2)

with \( \mathbf{E}(\mathbf{r}_o) \) the complex amplitude of the electric field at the location \( \mathbf{r}_o \) of the dipole. This field is the field by the dipole itself, the source field, and, for instance, the field reflected by an interface. If we only consider the source field, expression (5.2) reduces to \( P_o \) from Equation (5.1) [10]. When the particle is located near an interface, the reflected electric field adds to \( \mathbf{E}(\mathbf{r}_o) \), and the power acquires an additional term due to the reflected field.

We shall consider the power emitted by an electric dipole near the surface of an epsilon-near-zero (ENZ) material. Such materials are usually metamaterials. These artificial media are sub-wavelength structures that effectively act as a continuum for the frequency range under consideration. By tailoring the structures, in principal any values of the (relative) permittivity \( \varepsilon \) and (relative) permeability \( \mu \) can be obtained. An ENZ medium is a metamaterial with \( \varepsilon = 0 \) and \( \mu = 1 \). The first demonstrations of ENZ media were in the microwave and terahertz regions [11-15], and later ENZ materials for the visible part of the spectrum were constructed [16-18]. ENZ materials are nearly impenetrable for radiation. When a plane wave is incident upon the interface, the wave in the material is evanescent, and does not propagate into the medium [19]. An exception is when the wave is under (near) normal incidence. Then an oscillating electric field penetrates the material, but it has no spatial dependence. This phenomenon is called “static optics”. It can be used for letting radiation tunnel (or funnel) through the material without loss of phase information [20-
It also allows for the construction of an angular filter [26-29]. Another prediction is that the force between the dipole and the ENZ surface is repulsive [30-33], leading to possible levitation of the particle. We shall show that the presence of an ENZ interface greatly affects the power emission rate. An interesting aspect of this problem is that it can be solved analytically, whereas for other media one has to resort to a numerical approach.

5.2 Electric dipole near an interface

We first consider the more general case of an electric dipole located a distance $H$ away from the interface with a material, as illustrated in Figure 5.1. The dipole is located on the $z$ axis, and we take the surface as the $xy$ plane. The dipole is embedded in a medium with permittivity $\varepsilon_1$ and permeability $\mu_1$, both assumed to be positive. That assumption is not really a necessary restriction at this point, but we shall see below that it provides a huge computational advantage. The index of refraction is then $n_1 = \sqrt{\varepsilon_1 \mu_1}$. The medium can be a half-infinite material, or a more complicated stratified structure. We shall for now only assume that the reflection of a plane wave can be accounted for by the Fresnel reflection coefficients $R_s$ and $R_p$ for $s$ and $p$ polarized radiation, respectively.
The dipole is located a distance $H$ below an interface with a material medium. Partial waves in the angular spectrum can be traveling, indicated by arrows, or evanescent in the $z$ direction, indicated by dashed lines. Due to boundary conditions, each wave vector must have the same parallel component $k_\parallel$. The transmitted wave vector shown is for a semi-infinite medium, but that is only for illustration.

The electric field, emitted by the dipole, is the source field, and it can be represented by an angular spectrum of plane waves. We have [34]

$$
E_s(r) = \frac{i\mu_1 k_o d_o}{8\pi^2 \varepsilon_o} \sum_{\sigma=s,p} \int d^2k_\parallel \frac{e^{ik_\parallel \cdot r}}{v_1} (\hat{u} \cdot e_{\sigma,i}) e_{\sigma,i} e^{i\nu_1 (h+k_o z)} , -H < z < 0 .
$$

(5.3)

We have set $h = k_o H$ for the dimensionless distance between the particle and the surface. For each $k_\parallel$ this is a plane wave, and the integral runs over the $k_\parallel$ plane, which is the $xy$ plane. The wave vector of a partial wave is $k_i = k_\parallel + k_o v_1 e_z$, and these waves are the incident plane waves on the surface. Here, $v_1$ is the dimensionless $z$ component of the wave vector. First, we set
\[ \frac{k_{\parallel}}{k_0}, \quad (5.4) \]

for the dimensionless magnitude of \( k_{\parallel} \). From the dispersion relation it then follows that

\[
v_1 = \begin{cases} 
\sqrt{n_1^2 - \alpha^2}, & 0 \leq \alpha < n_1 \\
 i \sqrt{\alpha^2 - n_1^2}, & n_1 < \alpha < \infty
\end{cases}. \quad (5.5)
\]

For \( \alpha < n_1 \), the \( z \) component of the incident wave vector is real, and the plane wave is a traveling wave. With \( \theta_i \) the angle of incidence, we have \( \alpha = n_1 \sin \theta_i \). For \( \alpha > n_1 \), the \( z \) component of the incident wave is positive imaginary, and the wave decays exponentially in the positive \( z \) direction, which is the direction towards the surface. These are the evanescent waves of the incident field. This is schematically depicted in Figure 5.1. For \( \alpha = 0 \), we have normal incidence, and for \( \alpha \leq n \), we have grazing incidence. Borderline is \( \alpha = n_1 \), for which \( v_1 = 0 \). We then get a division by zero on the right-hand side of Equation (5.3). We shall see below that this singularity is integrable, and can be transformed away by a proper change of variables. The polarization vectors of an incident wave are

\[
e_{s,i} = e_z \times \hat{k}_{\parallel}, \quad (5.6)
\]

\[
e_{p,i} = \frac{1}{n_1} (\alpha e_z - v_1 \hat{k}_{\parallel}), \quad (5.7)
\]
given \( \hat{k}_\parallel = \mathbf{k}_\parallel / k_\parallel \), the unit vector in the \( \mathbf{k}_\parallel \) direction. The corresponding magnetic field follows, in general, from the electric field as

\[
\mathbf{B}(\mathbf{r}) = -\frac{i}{\omega} \nabla \times \mathbf{E}(\mathbf{r}) .
\] (5.8)

The field from Equation (5.3), together with the corresponding magnetic field, is the incident field on the interface. The reflection of each partial wave can be accounted for by a Fresnel reflection coefficient. We thus find the angular spectrum of the reflected electric field to be

\[
\mathbf{E}_r(\mathbf{r}) = \frac{i\mu_1 k_0 d_0}{8\pi^2 \varepsilon_o} \sum_{\sigma=s,p} \int d^2k_\parallel \frac{e^{ik_1r}}{\nu_1} (\hat{\mathbf{u}} \cdot \mathbf{e}_\sigma)(e_{\sigma,\nu_1} R_{\sigma} e^{i\nu_1(h_1-k_0z)}) .
\] (5.9)

The wave vector of a partial reflected wave is \( \mathbf{k}_r = \mathbf{k}_\parallel - k_0 \nu_1 \mathbf{e}_z \), which only differs from \( \mathbf{k}_i \) in the sign of its \( z \) component. If the incident wave is traveling, then so is the reflected wave. When the incident wave is evanescent, so is the reflected wave, but this wave decays into the direction away from the surface, as it should be. The polarization vector, \( \mathbf{e}_{s,r} \) is the same as \( \mathbf{e}_{s,i} \) from Equation (5.6). For \( p \) polarization, vector \( \mathbf{e}_{p,r} \) follows from Equation (5.7) by replacing \( \nu_1 \) by \(-\nu_1\). The reflected magnetic field follows from \( \mathbf{E}(\mathbf{r}) \) as in Equation (5.8).

5.3 Power emission

The emitted power by the dipole is given by Equation (5.2), with \( \mathbf{E}(\mathbf{r}_o) = \mathbf{E}_s(\mathbf{r}_o) + \mathbf{E}_r(\mathbf{r}_o) \), and \( \mathbf{r}_o = -Hz e_z \). For the power we write
\[ P = P_s + P_r , \quad (5.10) \]

in obvious notation. The power due to the source field cannot be directly computed from the expression for \( E_s(r) \) in Equation (5.3), since this expression only holds for \(-H < z < 0\). A much simpler approach is given in Ref. 10. We then immediately find

\[ P_s = \mu_1 n_1 P_0 , \quad (5.11) \]

with \( P_0 \) the emitted power in free space (Equation (5.1)). The embedding medium gives an extra factor of \( \mu_1 n_1 \).

To compute \( P_r \), we set \( r_o = -He_z \) in Equation (5.9). An immediate simplification is that \( \exp(i k || r_o) = 0 \). We then adopt polar coordinates \((k_\parallel, \phi)\) in the \( k_\parallel \) plane, and we change variables from \( k_\parallel \) to \( \alpha \) as in Equation (5.4). We then have \( k_\parallel = e_x \cos \phi + e_y \sin \phi \), and the polarization vectors can be expressed in terms of \( \alpha \) and \( \phi \). The Fresnel coefficients only depend on \( \alpha \). The integrals over \( \phi \) are then elementary. We split the dipole moment polarization vector \( \hat{u} \) in its perpendicular and parallel components with respect to the surface:

\[ \hat{u} = \hat{u}_\perp + \hat{u}_\parallel . \quad (5.12) \]

We then obtain for the emitted power

\[ P = \mu_1 n_1 P_0 \left[ (\hat{u}_\perp^* \cdot \hat{u}_\perp) w_\perp(h) + (\hat{u}_\parallel^* \cdot \hat{u}_\parallel) w_\parallel(h) \right] . \quad (5.13) \]

This expression involves the two functions.
\[ w_\perp(h) = 1 + \frac{3}{2n_1^3} \Re \left[ \int_0^\infty \frac{\alpha^3}{v_1} e^{2ihv_1 R_p(\alpha)} \right], \quad \text{(5.14)} \]

\[ w_\parallel(h) = 1 + \frac{3}{4n_1^3} \Re \left[ \int_0^\infty \frac{\alpha}{v_1} e^{2ihv_1 (n_1^2 R_s - v_1^2 R_p)} \right]. \quad \text{(5.15)} \]

The attractive result (5.13)-(5.15) holds for any dipole polarization \( \hat{u} \), and no use has been made of any properties of the reflection coefficients (apart from the fact that they are rotationally symmetric around the \( z \) axis, so they only depend on \( \alpha \)). The term “1” on the right-hand sides give \( P_s \) from Equation (5.11), since \( \hat{u}_\perp \cdot \hat{u}_\perp + \hat{u}_\parallel \cdot \hat{u}_\parallel = 1. \)

### 5.4 Traveling and evanescent contributions

The integration range \( 0 \leq \alpha < n_1 \) in Equations. (5.14) and (5.15) represents the contribution from the traveling incident waves in the angular spectrum. Similarly, the range \( \alpha > n_1 \) accounts for the evanescent dipole waves. We split the functions accordingly:

\[ w_\gamma(h) = 1 + w_\gamma(h)^{tr} + w_\gamma(h)^{ev}, \gamma = \perp, \parallel. \quad \text{(5.16)} \]

For the traveling waves, we make the substitution

\[ n_1 u = \sqrt{n_1^2 - \alpha^2}, (tr). \quad \text{(5.17)} \]

This yields the new representations
\[ w_{\perp}(h)^{tr} = \frac{3}{2} \text{Re} \left[ \int_0^1 du \, e^{i\beta u}(1 - u^2) R_p \right], \quad (5.18) \]

\[ w_\parallel(h)^{tr} = \frac{3}{4} \text{Re} \left[ \int_0^1 du \, e^{i\beta u}(R_s - u^2 R_p) \right]. \quad (5.19) \]

Here we introduced

\[ \beta = 2 n_1 h, \quad (5.20) \]

As compared to Equations. (5.14) and (5.15), the \(1/v_1\) singularity has disappeared, and so has the \(v_1\) in the exponent. The Fresnel coefficients are now evaluated at

\[ \alpha = n_1 \sqrt{1 - u^2 \ (tr)}. \quad (5.21) \]

For the evanescent contributions, we set

\[ n_1 u = \sqrt{\alpha^2 - n_1^2 \ (ev)}, \quad (5.22) \]

so that for the Fresnel coefficients we need to take

\[ \alpha = n_1 \sqrt{1 + u^2 \ (ev)}. \quad (5.23) \]

We then obtain the new representations
Interestingly, only the imaginary parts of the Fresnel coefficients come into these representations. We will see the significance of this for ENZ media in Sec. 5.9.

It is worthwhile noticing that the clean splitting in $tr + ev$ is a result of our assumption that there is no damping in the embedding medium. If $\varepsilon_1$ or $\mu_1$ would have an imaginary part, then $n_1$ would be complex, and so it would not be on the line of integration. Furthermore, we shall see in the next section that the traveling and evanescent parts of the emitted power have distinct physical interpretations.

### 5.5 Power through the interface

Conservation of energy implies that the emitted power $P$ by the dipole either radiates away into the region $z < 0$, where it ends up in the far field, or it passes through the surface. We shall write

$$ P = P_\perp + P_1, $$

and here $P_\perp$ represents the part that crosses the interface, whereas $P_1$ is the part that propagates to the far field in medium 1. In this section we derive an expression for $P_\perp$, without any assumptions yet about the material of the medium.
The flow of energy is accounted for by the time-averaged Poynting vector $\mathbf{S}(\mathbf{r})$. In the embedding medium, the region $z < 0$, this vector is

$$\mathbf{S}(\mathbf{r}) = \frac{1}{2\mu_0\mu_1} \text{Re}[\mathbf{E}(\mathbf{r})^* \times \mathbf{B}(\mathbf{r})].$$

(5.27)

The power passing through the surface $z = 0$ is then

$$P_\perp = \iint \mathbf{S}(x, y, 0) \cdot \mathbf{e}_z \, dA.$$  

(5.28)

With a vector identity this can be written as

$$P_\perp = \frac{1}{2\mu_0\mu_1} \text{Re}\left\{\iint \mathbf{E}(x, y, 0)^* \cdot [\mathbf{B}(x, y, 0) \times \mathbf{e}_z] \, dA\right\}.$$  

(5.29)

Both the electric and the magnetic fields are the sums of the source field and the reflected field. With Equations. (5.3) and (5.9) we find

$$\mathbf{E}(x, y, 0) = \frac{i\mu_1 k_0 d_0}{8\pi^2 \varepsilon_0} \sum_{\sigma=s,p} \int d\mathbf{k}_\parallel \frac{e^{ik_1 \cdot \mathbf{r}}}{\nu_1} e^{iv_1 h} (\hat{\mathbf{u}} \cdot \mathbf{e}_{\sigma,i}) [\mathbf{e}_{\sigma,i} + R_\sigma(\alpha)\mathbf{e}_{\sigma,r}] .$$

(5.30)

The magnetic field follows from Equation (5.8), and after working out the cross products between $\mathbf{e}_z$ and the polarization vectors we find

$$\mathbf{B}(x, y, 0) \times \mathbf{e}_z = \frac{i\mu_1 k_0 d_0 n_1}{8\pi^2 \varepsilon_0 c} \int d\mathbf{k}_\parallel \frac{e^{ik_1 \cdot \mathbf{r}}}{\nu_1} e^{iv_1 h}$$

(5.31)
\[
\times \left\{ \frac{v_1}{n_1} (\hat{\mathbf{u}} \cdot \mathbf{e}_{s,t}) \left[ 1 - R_s(\alpha) \right] \mathbf{e}_{s,t} - (\hat{\mathbf{u}} \cdot \mathbf{e}_{p,t}) \left[ 1 + R_p(\alpha) \right] \hat{\mathbf{k}}_\parallel \right\}.
\]

The right-hand sides of Equations (5.30) and (5.31) are then substituted into Equation (5.9). We are then left with a product of two angular spectra, which may seem daunting. However, the dependence on \(x\) and \(y\) only enters through the exponentials \(\exp \left( i \mathbf{k}_\parallel \cdot \mathbf{r} \right)\), and orthogonality comes to the rescue:

\[
\iint e^{i(k_\parallel - k'_\parallel) \cdot \mathbf{r}} \, dA = 4\pi^2 \delta(\mathbf{k}_\parallel - \mathbf{k}'_\parallel).
\]

When integrated over the \(xy\) plane, there is no coupling between different \(\mathbf{k}_\parallel\) modes of \(\mathbf{E}\) and \(\mathbf{B}\).

As in Sec. 3, we adopt polar coordinates \((k_\parallel, \phi)\) in the \(\mathbf{k}_\parallel\) plane. The \(\phi\) dependence only enters through the various polarization vector, and the integrations over \(\phi\) are easy. The result simplifies considerably:

\[
P_\perp = \frac{3 \mu_1}{8 n_1^2} P_0 \text{Re} \int_0^\infty d\alpha \frac{\alpha}{v_1} \left[ \frac{1}{2} \right] \left[ n_1^2 [1 + R_s(\alpha)] [1 - R_s(\alpha)'] (\hat{\mathbf{u}}_\perp \cdot \hat{\mathbf{u}}_\perp) + [1 + R_p(\alpha)] [1 - R_p(\alpha)'] \left( \mathbf{u}_\parallel^* \cdot \mathbf{u}_\parallel \right) + 2\alpha^2 \left( \mathbf{u}_\perp^* \cdot \mathbf{u}_\perp \right) \right].
\]

At this point it is advantageous to split the \(\alpha\) integral in its traveling and evanescent parts. We make use of the fact that \(v_1\) is real for traveling waves and imaginary for evanescent waves, as shown in Equation (5.5). After some regrouping, we finally obtain

\[
P_\perp = \mu_1 n_1 P_0 \left\{ \frac{1}{2} + (\hat{\mathbf{u}}_\perp \cdot \hat{\mathbf{u}}_\perp) [w_\perp(h)^{ev} - z_\perp] + (\hat{\mathbf{u}}_\parallel \cdot \hat{\mathbf{u}}_\parallel) [w_\parallel(h)^{ev} - z_\parallel] \right\}.
\]
Here we introduced the functions

\[ z_\perp = \frac{3}{4} \frac{\alpha^3}{v_1} \int_{0}^{n_1} d\alpha \frac{\alpha^2}{v_1} |R_p(\alpha)|^2, \]  

(5.35)

\[ z_\parallel = \frac{3}{8} \frac{\alpha^3}{n_1^2} \int_{0}^{n_1} d\alpha \frac{\alpha^3}{v_1} |n_1^2 R_s(\alpha)|^2 + v_1^2 |R_p(\alpha)|^2 \]  

(5.36)

The result (5.34) has an interesting interpretation. The \( \frac{1}{2} \) gives half of \( P_z \) from Equation (5.11). This term represents the fact that half the radiation that would be emitted without the interface is emitted towards the interface. The terms with \( z_\perp \) and \( z_\parallel \) represent the part that is reflected back into the region \( z < 0 \). They give a negative contribution, and they only depend on the absolute value of the Fresnel reflection coefficients. These functions are independent of \( h \), and they only contain traveling waves, as can be seen from the integration limits. The functions \( w_\perp \) and \( w_\parallel \) represent interference between source waves and reflected waves. Oddly enough, only the evanescent parts of these functions contribute to the transmission of power through the interface. This also implies that the traveling parts of these functions represent emitted power that travels directly to the far field in \( z < 0 \).

For the functions \( z_\perp \) and \( z_\parallel \) we can make the same change of variables as in Equation (5.17). This yields the simpler-looking forms

\[ z_\perp = \frac{3}{4} \int_{0}^{1} du (1 - u^2) |R_p(\alpha)|^2, \]  

(5.37)

\[ z_\parallel = \frac{3}{8} \int_{0}^{1} du [ |R_s(\alpha)|^2 + v_1^2 |R_p(\alpha)|^2 ] . \]  

(5.38)
The Fresnel coefficients are here evaluated at $\alpha$ from Equation (5.21).

### 5.6 Emitted power near an ENZ interface

We now consider the case where the medium is an ENZ material. We then have $\varepsilon = 0$, $\mu = 1$, and we shall also assume $\mu_1 = 1$ for the embedding medium. The Fresnel reflection coefficients for an ENZ medium are [19]

$$ R_s(\alpha) = \frac{1}{n_1^2} (v_1 - i\alpha)^2, \quad (5.39) $$

$$ R_p = -1, \quad (5.40) $$

with $v_1$ given by Equation (5.5). For a traveling wave we have $\alpha = \sin \theta_i$, with $\theta_i$ the angle of incidence. Then $v_1 = n_1 \cos \theta_i$, and $v_1 - i\alpha = n_1 \exp (-i\theta_i)$. Therefore

$$ R_s(\alpha) = e^{-2i\theta_i} (tr), \quad (5.41) $$

and we have $|R_s(\alpha)| = 1$. For evanescent waves, $R_s(\alpha)$ is real and limited by

$$ -1 < R_s(\alpha) < 0. \quad (5.42) $$

Figure 5.2 represents $R_s(\alpha)$ pictorially in the complex plane.
Figure 5.2  Shown is the reflection coefficient for \( s \) waves.

For traveling waves, \( R_s(\alpha) \) lies on the unit circle, and its phase angle is twice the angle of incidence, measured clockwise. For evanescent waves, \( R_s(\alpha) \) is negative.

For evanescent waves, \( R_s \) and \( R_p \) are real, and with Equations (5.24) and (5.25) we immediately find

\[
w_\perp(h)_{\text{ev}} = w_\parallel(h)_{\text{ev}} = 0.
\]  \hspace{1cm} (5.43)

Evanescent waves do not contribute to the emitted power. With \( R_p = -1 \), the integral in Equation (5.18) can be computed, and we find

\[
w_\perp(h) = 1 + \frac{3}{\beta^2} \left( \cos \beta - \frac{1}{\beta} \sin \beta \right).
\]  \hspace{1cm} (5.44)
Figure 5.3  The figure shows the two functions that determine the power emission near an ENZ interface, for $n_1 = 1$.

For the integral in Equation (5.19), we use Equation (5.39) for $R_s$, and we then obtain

$$w_\parallel(h) = 1 + \frac{3}{2\beta} \left[ \sin \beta + \frac{3}{\beta} \left( \cos \beta - \frac{1}{\beta} \sin \beta \right) \right] + \frac{3\pi}{4\beta} J_2(\beta),$$

(5.45)

with $J_2(\beta)$ a Bessel function. Expressions (5.44) and (5.45) are parameter free functions of $\beta$, and the graphs are shown in Figure 5.3 as a function of $h$. Both $w_\perp(h)$ and $w_\parallel(h)$ level off to unity for $h$ large, as it should be, since for $h$ large the effect of the interface should disappear.
5.7 Behavior for $h$ small

Expressions (5.44) and (5.45) seem to have negative powers of $\beta$, which suggests that these functions diverge for $\beta$, or $h$, small. Figure 5.3, however, shows that the functions are finite for $h = 0$. In order to study the behavior for $\beta$ small, we set $R_p = -1$ in Equation (5.18), which gives

$$w_{\perp}^{\text{tr}}(h) = -\frac{3}{2} \int_0^1 du \ (1 - u^2) \cos(\beta u) . \quad (5.46)$$

Then we expand $\cos(\beta u)$ in a series, and integrate term-by-term. The first term in the series cancels the 1 on the right-hand side of Equation (5.16), and we find

$$w_{\perp}(h) = -6 \sum_{k=1}^{\infty} \frac{k + 1}{(2k + 3)!} (-\beta^2)^k . \quad (5.47)$$

The first few terms are

$$w_{\perp}(h) = \frac{1}{10} \beta^2 - \frac{1}{280} \beta^4 + \cdots , \quad (5.48)$$

and in particular $w_{\perp}(0) = 0$. With some more effort, we find

$$w_\parallel(h) = 1 + 6 \sum_{k=1}^{\infty} \frac{k(k + 1)}{(2k + 3)!} (-\beta^2)^k + 3\pi\beta \sum_{k=0}^{\infty} \frac{1}{4^{k+2}k! (k+2)!} (-\beta^2)^k , \quad (5.49)$$

with the first few terms being
\[ w_\parallel(h) = 1 + \frac{3\pi}{32}\beta - \frac{1}{10}\beta^2 + \cdots, \quad (5.50) \]

for which \( w_\parallel(0) = 1 \), as in the graph.

### 5.8 Power crossing the ENZ interface

An interesting question is whether or not any energy passes through the interface. The general expression for \( P_\perp \) is given by Equation (5.34). We already found \( w_\perp(h)^{ev} = w_\parallel(h)^{ev} = 0 \) for the ENZ medium, so it remains to find \( z_\perp \) and \( z_\parallel \), defined by Equation (5.35) and (5.36). Only traveling waves contribute, and therefore we have \( |R_s| = 1 \) and \( |R_p| = 1 \). From the representations (5.37) and (5.38) we readily find

\[ z_\perp = z_\parallel = \frac{1}{2}, \quad (5.51) \]

and therefore

\[ P_\perp = 0. \quad (5.52) \]

No energy is transferred through the interface into the ENZ material.

It should be noted, however, that this does not necessarily imply that no energy penetrates the material. It can very well be that energy crosses into the material locally, but then returns somewhere close by back to the region \( z < 0 \). Such sub-wavelength back-and-forth oscillations of energy have been predicted for a regular dielectric-dielectric interface when the medium is thinner.
that the embedding medium of the dipole. The fact that $P_{\perp}$ vanishes only implies that the net energy flow across the interface is zero.

5.9 Role of damping

A medium with $\varepsilon$ exactly equal to zero is most likely not possible to construct with metamaterial technology. The functions $w_{\perp}(h)$ and $w_{\parallel}(h)$ shown in Figure 5.3 are universal functions representing the power emission near an ENZ interface, so for $\varepsilon = 0$. For arbitrary $\varepsilon$ these functions can be obtained by numerical integration of the representations given in Equations (5.18, 5.19, 5.24, 5.25). A typical example of $w_{\perp}(h)$ is shown by the solid curve in Figure 5.4 for $\varepsilon = 0.1i$. The dashed curve is the same as the solid curve in Figure 5.3. The most striking feature is that $w_{\perp}(h)$ diverges for $h \to 0$ for $\varepsilon \neq 0$.

The traveling part $w_{\perp}(h)$ is given by Equation (5.18). The integral is over a finite range, and is therefore finite. This implies that the diverging behavior for $h$ small has to come from the evanescent contribution from Equation (5.24). With $\varepsilon$ the permittivity of the medium and $\varepsilon_1$ the permittivity of the embedding medium of the dipole, the Fresnel reflection coefficient for $p$ waves is given by Equation (5.5), and since we are considering the evanescent range, $\alpha > n_1$, this parameter is positive imaginary. The parameter $\nu$ is defined similarly

$$R_p = \frac{\varepsilon \nu_1 - \varepsilon_1 \nu}{\varepsilon \nu_1 + \varepsilon_1 \nu}.$$  (5.53)

$$\nu = \sqrt{n^2 - \alpha^2}.$$  (5.54)

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Here, \( n = \sqrt{\varepsilon} \) is the index of refraction of the material, which is complex, in general. This \( \nu \) is the dimensionless \( z \) component of the plane wave that is transmitted into the medium. For an ENZ material we have \( \varepsilon = 0 \), and so \( \nu = i\alpha \), and this makes \( R_p \) real. Therefore, \( \text{Im}[R_p] = 0 \), and \( w_\perp(h)^{\text{ev}} = 0 \).

Figure 5.4 Shown is the function \( w_\perp(h) \) for \( \varepsilon = 0.1i \) (solid curve) and \( \varepsilon = 0 \) (dashed curve).

Let us now consider what happens when \( \varepsilon \neq 0 \). In Equation (5.54), \( \alpha > n_1 \), since we are considering the region where the incident waves are evanescent. Assume first that \( \varepsilon \) is real and positive, like for an ordinary dielectric. If \( \varepsilon < \varepsilon_1 \), then \( n < n_1 \), and \( v_1 \) is positive imaginary, since \( \alpha > n_1 \). Then transmitted wave is evanescent, \( R_p \) is real, \( \text{Im}[R_p] > 0 \), and \( w_\perp(h)^{\text{ev}} = 0 \). For \( \varepsilon > \varepsilon_1 \) we introduce the parameter

\[
u_0 = \frac{\varepsilon}{\sqrt{\varepsilon_1} - 1} ,
\]

(5.55)
which is positive. For $0 \leq u < u_o$ the transmitted wave is traveling, $v$ is real, $|R_p| = 1$, $\text{Im}[R_p] \neq 0$. For $u > u_o$, the transmitted wave is evanescent, and $\text{Im}[R_p] = 0$. So, in this case the integral over $u$ in Equation (5.24) runs to $u_o$, rather than infinity, and this gives a finite contribution to $w_\perp(h)^{ev}$.

Apparently, when $\varepsilon > 0$ the contribution of $w_\perp(h)^{ev}$ to $w_\perp(h)$ is either zero or finite. We now consider the situation where $\varepsilon$ has a positive imaginary part, responsible for damping in the material. Then $\text{Im}[R_p] \neq 0$ for all $u$, and the integral in Equation (5.24) runs to infinity. The part $u^2 \exp(-\beta u)$ of the integrand has a maximum at $u = 2/\beta$, and that maximum is $4/(\varepsilon \beta)^2$. For $\beta$ (and $h$) small, this maximum moves to high $u$ values, and the peak height increases as $1/\beta^2$. This function is multiplied $\text{Im}[R_p]$. This function is shown in Figure 5.5 for $\varepsilon_1 = 2$ and $\varepsilon = 5 + 2i$. We see that $\text{Im}[R_p]$ levels off to a constant, already for moderate values of $u$.

Since $u^2 \exp(-\beta u)$ gives the main contribution from the region $u \sim 2/\beta$, we can replace $\text{Im}[R_p(u)]$ by $\text{Im}[R_p(\infty)]$ as a good approximation for $\beta$ small. For $u$ large, we have $v = v_1$, and we find with Equation (5.51)

$$R_p(\infty) = \frac{\varepsilon - \varepsilon_1}{\varepsilon + \varepsilon_1}. \quad (5.56)$$

Taking the imaginary part and using that $\varepsilon$ is small then gives

$$\text{Im}[R_p(\infty)] = \frac{2}{\varepsilon_1} \text{Im}[\varepsilon]. \quad (5.57)$$
Then the integral in Equation (5.24) is easily computed. For $\text{Im}[\varepsilon] \neq 0$ this is then the dominant term for $\beta$ small. We thus obtain

$$w_\perp(h) = \frac{6}{\varepsilon_1 \beta^3} \text{Im}[\varepsilon] + \cdots,$$

(5.58)

and along similar lines we find

$$w_\parallel(h) = \frac{3}{\varepsilon_1 \beta^3} \text{Im}[\varepsilon] + \cdots.$$

(5.59)

Figure 5.6 shows $w_\perp(h)$ for $\varepsilon_1 = 0.5$ and $\varepsilon = 0.1i$. This is the solid curve, obtained by numerical integration. The dashed curve is the approximation by the term shown on the right-hand side of Equation (5.58). We see from the graph that this approximation is excellent for small values of $h$. This also shows that $w_\perp(h)$ will always be dominated by this diverging term for $\beta$ small, no matter how small the imaginary part of $\varepsilon$ is. For decreasing values of $\text{Im}[\varepsilon]$, the curve moves closer to the vertical axis, but only for $\text{Im}[\varepsilon] \equiv 0$ do we get $w_\perp(0) = 0$, as in Figure 5.3.
Figure 5.5 The graph shows the imaginary part of $R_p$ as a function of $u$ for $\varepsilon_1 = 2$ and $\varepsilon = 5 + 2i$.

Figure 5.6 The graph illustrates that the approximation of $w_\perp(h)$ by the term shown on the right-hand side of Equation (5.58) is excellent at small distances.

5.10 Conclusions

The power emitted by an electric dipole embedded in a medium is given by $P_s$ in Equation (5.11). When an interface is present, some radiation is reflected back to the dipole, and this gives an induced power emission $P_r$. The total emitted power $P$ is given by Equation (5.13) in terms of the
functions $w_\perp(h)$ and $w_\parallel(h)$ from Equations (5.14) and (5.15). These functions depend explicitly
on the dimensionless distance $h$ between the dipole and the interface, and implicitly on the material
parameters of the medium through the Fresnel reflection coefficients $R_s$ and $R_p$ for $s$ and $p$
polarized plane waves, respectively. It is advantageous to split these functions in contributions
from traveling and evanescent incident plane waves, as shown in Sec. 5.4. Then the singularity $\frac{1}{\nu_1}$
can be removed by appropriate changes of variables. Also, when considering the total power
transmitted through the interface, only the evanescent parts of $w_\perp(h)$ and $w_\parallel(h)$ contribute, as
shown in Equation (5.34).

The functions $w_\perp(h)$ and $w_\parallel(h)$ can be obtained in closed form for an ENZ interface, as
shown in Sec. 6. Only the traveling parts of these functions contribute to the power emission, and
the total power crossing the interface is zero. In Sec. 5.9 we have shown that even the smallest
imaginary part in $\varepsilon$ of the ENZ material leads to a diverging behavior of the power when the
distance between the dipole and the surface becomes small. This is due to the contribution from
the evanescent waves, which is zero for a pure ENZ material with $\varepsilon = 0$. 
5.11 References


An oscillating magnetic dipole moment emits radiation. We assume that the dipole is embedded in a medium with relative permittivity $\varepsilon_r$ and relative permeability $\mu_r$ and we have studied the effects of the surrounding material on the flow lines of the emitted energy. For a linear dipole moment in free space the flow lines of energy are straight lines, coming out of the dipole. When located in a medium, these field lines curve toward the dipole axis, due to the imaginary part of $\mu_r$. Some field lines end on the dipole axis, giving a non-radiating contribution to the energy flow. For a rotating dipole moment in free space, each field line of energy flow lies on a cone around the axis perpendicular to the plane of rotation of the dipole moment. The field line pattern is an optical vortex. When embedded in a material, the cone shape of the vortex becomes a funnel shape, and the windings are much less dense than for the pattern in free space. This is again due to the imaginary part of $\mu_r$. When the real part of $\mu_r$ is negative, the field lines of the vortex swirl around the dipole axis opposite to the rotation direction of the dipole moment. For a near-single-negative medium, the spatial extent of the vortex becomes huge. We compare the results for the magnetic dipole to the case of an embedded electric dipole.
6.1 Introduction

The optical properties of a linear, homogeneous, isotropic material are represented by the relative permittivity $\varepsilon_r$ and the relative permeability $\mu_r$. Both parameters are complex, in general, with a nonnegative imaginary part. The index of refraction $n$ is defined as

$$n^2 = \varepsilon_r \mu_r , \text{Im}[n] \geq 0 .$$

(6.1)

This leaves an ambiguity if $\varepsilon_r$ and $\mu_r$ are both positive or both negative. Then we include small positive imaginary parts in these parameters, and consider the limit where these imaginary parts approach zero. We then find that $n$ is positive when $\varepsilon_r$ and $\mu_r$ are both positive (normal dielectric material) and $n$ is negative when $\varepsilon_r$ and $\mu_r$ are both negative (negative index of refraction material).

When a plane wave of light travels through a medium, the wavelength changes, as compared to propagation in free space, and this is due to the real part of $n$. The effect of the imaginary part of $n$ is damping of the amplitudes of the electric and magnetic fields in the propagation direction, and this gives a corresponding damping of the intensity along the direction of propagation. The disappearing energy is absorbed by the material. The flow lines of energy are the field lines of the Poynting vector. For propagation in free space, these field lines are straight, and they remain straight for propagation in a material. One could argue that the damping only affects the magnitude of the Poynting vector, and, since field lines are only determined by the direction of the Poynting vector, the field lines should remain unaltered for propagation in a medium. The damping, due to absorption by the material, diminishes the intensity along the propagation direction, but it does not affect the paths of energy flow. For a plane wave, this argument holds true, but in general the effect of damping is more intricate, as we shall show below.
6.2 Magnetic dipole radiation

We consider a magnetic dipole oscillating at angular frequency \( \omega \) and located at the origin of coordinates. The dipole moment is given by

\[
p(t) = \text{Re}\left[pe^{-i\omega t}\right],
\]

(6.2)

with \( p \) being the complex amplitude. The dipole is embedded in a medium with relative permittivity \( \varepsilon_r \) and relative permeability \( \mu_r \). The emitted electric field is time harmonic,

\[
E(r, t) = \text{Re}\left[E(r)e^{-i\omega t}\right],
\]

(6.3)

with \( E(r) \) being the complex amplitude, and a similar representation holds for the magnetic field \( B(r, t) \). With a slight generalization of [1] we obtain,

\[
E(r) = \frac{\mu_r n k_0^2}{4\pi \varepsilon_o c} (p \times \hat{r}) \left(1 + \frac{i}{nk_0 r}\right) \frac{e^{i n k_0 r}}{r},
\]

(6.4)

\[
B(r) = \frac{n \mu_r n k_0^2}{c \varepsilon_o c} \left[p - (p \cdot \hat{r})\hat{r} + \left[p - 3(p \cdot \hat{r})\hat{r}\right]\right]
\]

\[
\times \frac{i}{nk_0 r} \left(1 + \frac{i}{nk_0 r}\right) \frac{e^{i n k_0 r}}{r},
\]

(6.5)

with \( k_0 = \omega / c \), \( \hat{r} = r / r \), and \( r \neq 0 \).

In order to simplify the notation, we set \( p = p_o \hat{u} \), with \( p_o > 0 \) and \( \hat{u} \) normalized as \( \hat{u} \cdot \hat{u}^* = 1 \). The overall constant is abbreviated as

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\[ \zeta = \frac{\mu_n k_0^3 p_0}{4 \pi \varepsilon_0 c}. \] (6.6)

The dimensionless distance between the dipole and the field point is indicated by \( q = k_o r \) and the dimensionless fields \( e(q) \) and \( b(q) \), with \( q = k_o r \), are defined as

\[
E(r) = \zeta e(q),
\]

\[
B(r) = \frac{\zeta}{c} b(q).
\]

We then obtain

\[
e(r) = (\hat{u} \times \hat{q})(1 + \frac{i}{nq}) \frac{e^{inq}}{q},
\]

\[
b(q) = n\left\{ \hat{u} - (\hat{u} \cdot \hat{q}) \hat{q} + [\hat{u} - 3(\hat{u} \cdot \hat{q}) \hat{q}] \frac{i}{nq} \left( 1 + \frac{i}{nq} \right) \right\} e^{inq} \frac{1}{q}.
\]

Here, \( \hat{q} = q/q = \hat{r} \). These dimensionless fields only depend on the dimensionless position vector \( q \) of the field point.

### 6.3 Poynting vector

The time-averaged Poynting vector for electromagnetic radiation in a linear, homogeneous, isotropic material is given by
\[ S(r) = \frac{1}{2\mu_0} \text{Re} \left[ \frac{1}{\mu_r} E(r)^* \times B(r) \right]. \quad (6.11) \]

For magnetic dipole radiation this can be expressed in terms of the dimensionless fields as

\[ S(q) = \frac{|\zeta|^2}{2\mu_0 c} \text{Re} \left[ \frac{1}{\mu_r} e(q)^* \times b(q) \right]. \quad (6.12) \]

We notice that \( S \) only depends on the field point \( r \) through the dimensionless position vector \( q \).

With expressions (6.9) and (6.10), the right-hand side of Equation (6.12) can be worked out. We split off an overall factor:

\[ S(q) = \frac{|\zeta|^2}{2\mu_0 c} \frac{1}{q^2} e^{-2q\text{Im}[n]} \sigma(q). \quad (6.13) \]

This defines the dimensionless vector \( \sigma(q) \). The positive overall factor depends on the field point through \( q \). Since field lines of a vector field only depend on the direction of the vectors, the vector field \( \sigma(q) \) has the same field lines as the vector field \( S(q) \). We shall refer to \( \sigma(q) \) as the Poynting vector.

We obtain explicitly

\[ \sigma(q) = [1 - (\hat{u} \cdot \hat{q})(\hat{u}^* \cdot \hat{q})]q \text{Re} \left[ \frac{n}{\mu_r} \left( 1 + \frac{i}{nq} \right)^* \right] \]
\[ + \frac{1}{q|\mu_r|^2} \left\{ [1 - 3(\hat{u} \cdot \hat{q})(\hat{u}^* \cdot \hat{q})]q \text{Im}[\mu_r] + 2\text{Im}[\mu_r(\hat{u}^* \cdot \hat{q})]\hat{u} \right\}. \quad (6.14) \]
In terms of the dimensionless coordinate $q$, a distance of $2\pi$ corresponds to one free-space optical wavelength. The far field (many wavelengths from the source) is therefore the region $q \gg 1$. Equation (6.14) then simplifies to

$$\sigma(q) \approx [1 - (\hat{u} \cdot \hat{q})(\hat{u}^* \cdot \hat{q})] \hat{q} \text{Re} \left( \frac{n}{\mu_r} \right).$$  \hspace{1cm} (6.15)$$

The factor $[1 - (\hat{u} \cdot \hat{q})(\hat{u}^* \cdot \hat{q})]$ is positive (except may be zero for a certain direction $\hat{q}$), and it can be shown that [2],

$$\text{Re} \left[ \frac{n}{\mu_r} \right] \geq 0.$$  \hspace{1cm} (6.16)$$

The equal sign only holds for $\varepsilon_r > 0$ and $\mu_r < 0$, or $\varepsilon_r < 0$ and $\mu_r > 0$. Such materials are called single-negative, and we shall exclude this case for now. We come back to this interesting case in Section 6. The Poynting vector $\sigma(q)$ in the far field is therefore approximately a positive (or zero) constant times $\hat{q}$, and consequently the field lines are approximately straight, running outward from the dipole. Conversely, this implies that any curving of the field lines can only occur in the near field.

Let us now return to the general expression (6.14). It can be shown that

$$\text{Re} \left[ \frac{n}{\mu_r} \left( 1 + \frac{i}{nq} \right)^* \right] \geq 0,$$  \hspace{1cm} (6.17)$$

with the equal sign only holding for single-negative materials. Therefore, the first term on the right-hand side of Equation (6.14) is a positive (or zero) constant times $\hat{q}$, giving rise to radially
outward-running straight field lines. The first term in braces on the right-hand side of Equation (6.14) is also proportional to $\mathbf{q}$ (although the multiplying factor may not be positive), so this term also gives a radial contribution to the field lines. The second term in braces, $2\text{Im}[\mu_r(\mathbf{u}^* \cdot \mathbf{q})]\mathbf{u}$, is not proportional to $\mathbf{q}$, and therefore any curving of the field lines comes from this term.

It is interesting to notice that the material parameters in the expressions above only appear through $\mu_r$ and $\eta$. There is no explicit dependence on $\varepsilon_r$. Equation (6.14) in [3] gives the expression for $\sigma(\mathbf{q})$ for the case of an embedded electric dipole. The terms in braces are identical, with $\varepsilon_r$ and $\mu_r$ switched. For an electric dipole, however, there is also an explicit dependence on $\mu_r$.

The field lines of the Poynting vector are obtained as follows. Let $\mathbf{q}(u)$ be a parametrization of a field line. Any field line is the solution of $d\mathbf{q}(u)/du = \sigma(\mathbf{q}(u))$. We select an initial point with Cartesian dimensionless coordinates $(\bar{x}_o, \bar{y}_o, \bar{z}_o)$. Here, $\bar{x} = k_o x$, and so on. The field line through the selected point follows from integrating $d\mathbf{q}(u)/du = \sigma(\mathbf{q}(u))$ with $u = 0$ corresponding to the initial point. The direction of the field line is the direction of increasing $u$. The numerical integration is done with Mathematica.

### 6.4 Linear dipole

When the unit vector $\hat{\mathbf{u}}$ is real, we have $\mathbf{p}(t) = p_o \hat{\mathbf{u}} \cos(\omega t)$, and the dipole moment oscillates along the vector $\hat{\mathbf{u}}$. This is a linear dipole. We shall take $\hat{\mathbf{u}} = \mathbf{e}_z$, so the dipole moment oscillates along the $z$ axis. It is easy to verify that $\sigma(\mathbf{q})$ is rotation symmetric around the $z$ axis and reflection symmetric in the $xy$ plane. We therefore only consider field lines in the $yz$ plane, with $y \ge 0$ and $z \ge 0$. We have $\hat{\mathbf{u}} \cdot \mathbf{q} = \cos \theta$. The term $2\text{Im}[\mu_r(\mathbf{u}^* \cdot \mathbf{q})\mathbf{u}]$ becomes $2\mathbf{e}_z \cos \theta \text{Im}[\mu_r]$. This term is proportional to $\mathbf{e}_z$, whereas all other terms are proportional to $\mathbf{q}$. Moreover, this term is positive, and therefore the field lines will deviate from the radial direction.
such that they bend “up” toward the positive z axis. Furthermore, the term is proportional to \( \text{Im}[\mu_r] \), and therefore the curving is entirely due to the imaginary part of the permeability.

We find explicitly,

\[
\sigma(q) = \sin^2 \theta \, \hat{q} \, \text{Re} \left[ \frac{n}{\mu_r} \left( 1 + \frac{i}{nq} \right)^* \right] \\
+ \frac{1}{q|\mu_r|^2} \left| 1 + \frac{i}{nq} \right|^2 \left\{ [1 - 3 \cos^2 \theta] \hat{q} + 2 e_z \cos \theta \right\} \text{Im}[\mu_r].
\]  

(6.18)

Interestingly, the entire term in braces is proportional to \( \text{Im}[\mu_r] \). For free space we have \( \varepsilon_r = \mu_r = n = 1 \), and the Poynting vector becomes \( \sigma(q) = \sin^2 \theta \, \hat{q} \). The field line pattern is shown in Figure 6.1. The field lines are straight, coming out of the dipole. Figure 6.2 shows the field lines for \( \varepsilon_r = 2 \) and \( \mu_r = 1.5 + 1.8i \). The field lines bend up due to \( \text{Im}[\mu_r] \neq 0 \). We see from the figure that near the z axis the field lines end on the z axis rather than running to infinity. Consider a field point close to the z axis, so \( \bar{y} \) is small. Equation (18) becomes approximately
Figure 6.1  The figure shows the field lines of the Poynting vector for a magnetic dipole oscillating along the $z$ axis.

The field lines are straight. The shown density of the field lines has no significance.

Figure 6.2  Field lines for a linear magnetic dipole embedded in a medium with $\varepsilon_r$ and $\mu_r = 1.5 + 0.8i$ are curved.

The bending toward the $z$ axis is a consequence of the nonzero imaginary part of $\mu_r$.  

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\[
\sigma(q) \approx -2e_y \frac{\bar{y}}{q^2|\mu_r|} \left| 1 + \left( \frac{i}{nq} \right)^2 \Im[\mu_r] \right|.
\]

(6.19)

We see that the Poynting vector is into the negative \(y\) direction, and therefore every field line hits the \(z\) axis under 90°. The field lines end at the \(z\) axis. The \(z\) axis is a singular line, and \(\sigma(q) = 0\) on the \(z\) axis. Energy flowing along field lines that end at the \(z\) axis do not contribute to the radiated power. Similar behavior was found for an electric dipole [4], where the bending of the field lines resulted from the imaginary part of the permittivity. Figure 6.3 shows a larger view of the same graph as in Figure 6.2. We see that there is a cylindrical subwavelength region around the \(z\) axis that contains the field lines that end up on the \(z\) axis. Field lines outside this region run to infinity, and they level off to straight lines in the far field.

![Field line pattern](image)

Figure 6.3  The figure shows a larger view of the field line pattern of Figure 6.2.

Field lines close to the \(z\) axis bend toward the axis, hitting it perpendicularly. Other field lines run to infinity as almost straight lines.
6.5 Circular dipole

When we take vector \( \hat{u} \) as

\[
\hat{u} = -\frac{1}{\sqrt{2}}(e_x + ie_y),
\]

then the dipole moment \( \mathbf{p}(t) \) rotates in the \( xy \) plane with angular frequency \( \omega \). The rotation direction is counterclockwise when viewed down the positive \( z \) axis. We now have

\[
\hat{u} \cdot \hat{q} = -\frac{1}{\sqrt{2}} \sin \theta e^{i\phi},
\]

and the only term which is not proportional to \( \hat{q} \) becomes

\[
2 \text{Im}[\mu_r(\hat{u}^* \cdot \hat{q})\hat{u}] = \sin \theta [e_\rho \text{Im}(\mu_r) + e_\phi \text{Re}(\mu_r)].
\]

The unit vectors \( e_\rho \) and \( e_\phi \) are given by

\[
e_\rho = e_x \cos \phi + e_y \sin \phi,
\]

\[
e_\phi = -e_x \sin \phi + e_y \cos \phi,
\]

and therefore the right-hand side of Equation (6.22) is a vector in the \( xy \) plane. The Poynting vector from Equation (6.14) becomes
\[
\sigma(q) = \left[1 - \frac{1}{2}(\sin \theta)^2\right] \hat{q} \text{ Re}\left[\frac{n}{\mu_r} \left(1 + \frac{i}{nq}\right)^2\right] + \frac{1}{q|\mu_r|^2} \left|1 + \frac{i}{nq}\right|^2 \\
\left\{\left[1 - \frac{3}{2}(\sin \theta)^2\right] \hat{q} \text{ Im}[\mu_r] + \sin \theta \left[e_{\rho} \text{ Im}(\mu_r) + e_\phi \text{ Re}(\mu_r)\right]\right\} + \cdots. \quad (6.25)
\]

The Poynting vector is reflection symmetric in the \(xy\) plane, so we consider \(z \geq 0\) only.

For a circular magnetic dipole in free space we have

\[
\sigma(q) = \left[1 - \frac{1}{2}(\sin \theta)^2\right] \hat{q} + \frac{1}{q} \left(1 + \frac{1}{q^2}\right) \sin \theta \ e_\phi. \quad (6.26)
\]

This result is identical to the expression for an electric dipole in free space \([5]\). The basis vectors in a spherical coordinate system are \(\hat{q}, e_\theta,\) and \(e_\phi\). Vector \(\sigma(q)\) has no \(e_\theta\) component, and therefore \(\theta\) is constant along a field line. Each field line lies on a cone around the \(z\) axis. The \(e_\phi\) component is positive, and therefore the field lines wind around \(z\) axis into the direction of increasing \(\phi\). This is counterclockwise when viewed down the positive \(z\) axis, and so the rotation direction of the field lines around the \(z\) axis is the same as the rotation direction of the rotating dipole moment. Figure 6.4 shows a typical field line. Within about a wavelength from the location of the dipole, a field line winds around the \(z\) axis, and at a larger distance it levels off to approximately a straight line. The field line pattern has a vortex structure near the dipole, and the field lines run off in the radial direction in the far field. For a point on the positive \(z\) axis we have \(\theta = 0\), and the Poynting vector becomes \(\sigma(q) = e_z\). Therefore, the \(z\) axis is a field line.
Figure 6.4  Shown is a field line of the Poynting vector for a rotating dipole moment in free space.

The field line lies on a cone around the $z$ axis. Close to the dipole, it winds around the $z$ axis numerous times, and in the far field it levels off to approximately a straight line.

For radiation in a medium, the Poynting vector is given by Equation (6.25). It is easy to see that the $z$ axis is still a field line. Off the $z$ axis, the Poynting vector has a $\hat{q}$ component and an $\hat{e}_\phi$ component. Vector $\hat{e}_\rho$ can be expressed as

$$\hat{e}_\rho = \hat{q} \sin \theta + \hat{e}_\theta \cos \theta ,$$

(6.27)

so vector $\sigma(q)$ also has a $\hat{e}_\theta$ component. Therefore $\theta$ varies along a field line. The $\hat{e}_\phi$ component equals $\sin \theta \cos \theta \mathrm{Im}[\mu_r]$ multiplied by a positive function of $q$. For $z > 0$ this is positive, provided that $\mathrm{Im}[\mu_r] \neq 0$. Therefore, $\theta$ increases along a field line, and the cone shape for radiation in free space becomes a funnel shape. This is shown in Figure 6.5. The funnel shape is due to the imaginary part of $\mu_r$. For an electric dipole, a funnel shape appears due to the imaginary
part of $\varepsilon_r$. Another effect is that with an increasing of $\text{Im}[\mu_r]$, the windings around the $z$ axis become less dense. This is illustrated in Figure 6.6.

![Figure 6.5](image1)

Figure 6.5  The graph shows a field line of the Poynting vector for material parameters $\varepsilon_r = 1$ and $\mu_r = 0.8 + 0.01i$.

The cone shape of Figure 6.4 becomes a funnel shape due to $\text{Im}[\mu_r] \neq 0$.

![Figure 6.6](image2)

Figure 6.6  Shown is a field line of the Poynting vector for $\varepsilon_r = 1$ and $\mu_r = 0.8 + 0.1i$.

The only difference with the parameters for Figure 6.5 is that the imaginary part of $\mu_r$ is 10 times larger. As a result, the windings are much less tight.
The swirling of the field lines around the $z$ axis comes from the $e_{\phi}$ component of $\mathbf{\sigma}(q)$. This component is a positive function of $q$ time $\text{Re}[\mu_r]$. If $\text{Re}[\mu_r] > 0$, as for free space, the rotation direction is counterclockwise when viewed down the positive $z$ axis, and this is the same rotation direction as the magnetic dipole moment. If $\text{Re}[\mu_r] > 0$, however, the $e_{\phi}$ component of $\mathbf{\sigma}(q)$ is negative, and this reverses the rotation direction of the field lines around the $z$ axis. In this case, the flow of energy counter-rotates the rotation direction of the dipole moment. For an electric dipole, the same effects are attributed to $\text{Re}[\varepsilon_r]$.

### 6.6 Near-single-negative materials

For a circular dipole we have in the far field,

$$\mathbf{\sigma}(q) \approx \left[ 1 - \frac{1}{2}\sin^2\theta \right] \hat{q} \text{ Re} \left[ \frac{n}{\mu_r} \left( 1 + \frac{i}{nq} \right)^* \right],$$

(6.28)

which is proportional to $\hat{q}$. Therefore, the field lines run approximately radially outward, and they are approximately straight. A single-negative material has a real $\varepsilon_r$ and a real $\mu_r$, and they are of opposite sign. From Equation (6.1) we see that $n^2$ is negative, and so $n$ is positive imaginary. The expression in square brackets in Equation (6.28) is pure imaginary, so $\text{Re}[\cdots] = 0$. We conclude that for a single-negative material the right-hand side of Equation (6.28) vanishes. Apparently, for such materials the far-field term of the field line pattern is absent. Consequently, the remaining near-field terms continue to dominate in the far field. In the near field, the field lines swirl around the $z$ axis, and therefore we expect this pattern to extend into the far field. The vortex structure is not of subwavelength scale anymore, but extends into the far field. This leads to a huge vortex pattern, when seen on the scale of a wavelength.
A perfect single-negative material does not exist, so we consider near-single-negative materials, which have a small positive imaginary part in $\varepsilon_r$, in $\mu_r$, or in both. Figure 6.7 shows a field line of the Poynting vector for $\varepsilon_r = -1 + 0.1i$ and $\mu_r = 0.8$. The field line lies on a cone, because $\text{Im}[\mu_r] = 0$. As compared to Figure 6.4, we see that the spatial extent of the vortex is very large. If we make $\text{Im}[\varepsilon_r]$ smaller, the size of the vortex increases even more. Figure 6.8 shows a field line for $\varepsilon_r = 1$ and $\mu_r = -0.8 + 0.01i$. The shape is a funnel; the rotation direction is reversed, as compared to the rotation direction of the dipole moment; and the spatial extent of the vortex is immense. In fact, it is much larger than shown in the figure. Also interesting to see is that the field line flattens out. At some distance from the dipole, the field line appears to be rotating around the $z$ axis while remaining approximately in a plane parallel to the $xy$ plane. In the theoretical limit of a perfect single-negative material, the field lines are circles around the $z$ axis.

![Figure 6.7](image)

Figure 6.7  Shown is a field line of the Poynting vector for a rotating dipole. The material parameters are $\varepsilon_r = -1 + 0.1i$ and $\mu_r = 0.8$. 

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The material parameters $\varepsilon_r = 1$ and $\mu_r = -0.8 + 0.01i$.

6.7 Conclusions

A magnetic dipole is embedded in a medium with relative permittivity $\varepsilon_r$ and relative permeability $\mu_r$. We have studied the energy flow patterns of the emitted radiation as it propagates through the material. For a linear dipole, the field lines of energy flow bend toward the dipole axis due to the imaginary part of $\mu_r$. Field lines close to the axis end at the axis, whereas other field lines run to infinity. For a circular dipole, the field lines wind around the axis perpendicular to the plane of rotation in the near field. In the far field they level off to straight lines. In free space, each field line lies on a cone. The effect of the imaginary part of $\mu_r$ is that the cone shape becomes a funnel shape, and the windings are less dense than for the case of free space. When the real part of $\mu_r$ is negative, the rotation of the field lines around the axis is opposite to the rotation direction of the magnetic dipole moment. For a near-single-negative embedding medium, the spatial extent of the optical vortex becomes enormous.
6.8 References


CHAPTER VII
SUMMARY AND FUTURE

Metamaterial research has attracted scientists’ eyes in the recent years, yet many methods are developed to retrieve the fundamental information from the structure of the material. The interaction between the simple oscillating dipole with the mirror interface provides a promising tool in the new chapter of levitation of the dipole above the ENZ surface study. We briefly introduced the history of the interaction between the dipole and some interfaces, such as mirrors, ENZ surface. Major forces accounting for electric levitation are dependent on the material properties. When a harmonic oscillating dipole is located above the ENZ interface, there will be traveling waves at all angles, the evanescent waves also need to be considered. The superposition of the incident waves and reflected waves have more complicated forms, once we consider both the traveling waves and evanescent waves. However, it not only makes singularities and vortex in the energy flow field lines nearby, but also makes electric levitation possible.

We studied the energy flow patterns of the radiation emitted by an electric dipole located in between parallel mirrors. It appears that the field lines of the Poynting vector (the flow lines of energy) can have very intricate structures, including many singularities and vortices. The locations of the vortices and singularities can be found in an independent way. This provides a different means of studying the propagation of dipole radiation in between mirrors. A radiating electric dipole is considered, which is located near the joint of two orthogonal mirrors. The field lines of energy flow in the neighborhood of the dipole have an unique structure, depending on the state of
oscillation of the dipole and its distance to each mirror. There are numerous singularities and vortices in the sub-wavelength region between the dipole and the mirrors.

Secondly, we worked on the reflection, transmission and energy flow of a dipole’s radiation near an $\varepsilon$-near-zero material. The Fresnel reflection and transmission coefficients for $s$ and $p$ polarized light as a function of the angle of incidence in the ENZ limit were considered. For $p$ polarization we find that the reflection coefficient is $-1$ and the transmission coefficient is zero for all angles of incidence. This seems to imply that no radiation penetrates the material. More careful analysis shows that the electric field does penetrate the material. The transmitted electric field is evanescent and circularly polarized for all angles of incidence. The transmitted magnetic field is identically zero. For $s$ polarization, the transmitted electric field is $s$ polarized and the transmitted magnetic field is circularly polarized. The question we ask is the reflection and transmission of an electromagnetic plane wave of a dipole near an epsilon-near-zero (ENZ) medium. Besides, the field lines of the Poynting vector (the flow line of energy) of a dipole near an epsilon-near-zero (ENZ) medium is very different for $s$ and $p$ polarized waves.

Thirdly, we study some unique properties for the levitation or repulsion of a polarized dipole radiating at frequency, distance from an epsilon-near-zero (ENZ) metamaterial. There is a force acting on an electric dipole, which is located near an interface, by its own reflected radiation. A closed-form expression for this force was obtained for the case where the medium is an ENZ material. The result is always true for any state of oscillation or rotation of the dipole moment. The force is (mainly) perpendicular to the surface, and it is shown that, for close (sub-wavelength) distances between the dipole and the interface, this force is repulsive. It is shown that the force is exerted on the dipole by the reflected evanescent waves of the angular spectrum of the radiation.
Power emission by an electric dipole near an interface is considered. We derive explicit expressions for the emitted power for any state of oscillation of the dipole, without making use of the material properties of the substrate, and we derive an expression for the power crossing the interface. It is shown that the power naturally splits in contributions from traveling and evanescent incident waves. We then consider an ENZ material and obtain explicit expressions for the power. It is shown that only traveling waves contribute, and that no power crosses into the material.

Based on these demonstrated applications, a series of future works can be explored following this dissertation.

As recent research shows, materials with permittivity and permeability close to zero are not commonly available in nature. The epsilon-near-zero properties studied above have been achieved with both plasmonic and dielectric metamaterials. However, ENZ properties have only recently achieved in elaborate nanostructured three-dimensional metamaterials, within micron-size area samples. Besides that, ENZ properties are realized within anisotropic metamaterials with effective permittivity tensor \( \{\varepsilon_x, \varepsilon_y, \varepsilon_z\} \). We can consider a particle with an oscillating electric dipole located near an anisotropic substrate, acts a force if one of the components of permittivity tends to zero. We can analytically derive the force acted on the dipole by reflected waves, with both the traveling waves and evanescent waves.

For an ENZ metamaterial, we can also consider the quantum electromagnetic waves acting on it, especially how they modify the zero-point energy of the electromagnetic field and the resulting mechanical force of the quantum vacuum, the Casimir force.
APPENDIX A

SUPPLEMENT TO CHAPTER II
The complex amplitude of the electric field of each of the four dipoles is given by Equation (2.4), with the \( q \)'s given by Equations (2.6)-(2.9) and the \((\mathbf{u})\)'s are shown in Figure 2.2. The total field is the sum of the four. For a field point in the \( xy \) plane we have \( q_4 = q_1 \) and \( q_3 = q_2 \), and with some manipulations we find

\[
\mathbf{e} = 2e_z \left\{ \hat{u}_x + \frac{1}{q_1^2} (q_1 \cdot \hat{u}) h_z + \left[ \hat{u}_x + \frac{3}{q_1^2} (q_1 \cdot \hat{u}) h_z \right] \frac{i}{q_1} \frac{1 + i}{q_1} \right\} e^{iq_1} \\
- 2e_z \left\{ \hat{u}_x + \frac{1}{q_2^2} (q_1 \cdot \hat{u} - 2\tilde{y}\hat{u}_y) h_z \right\} \frac{i}{q_2} \frac{1 + i}{q_2} e^{iq_2} \\
+ \left[ \hat{u}_x + \frac{3}{q_2^2} (q_1 \cdot \hat{u} - 2\tilde{y}\hat{u}_y) h_z \right] \frac{i}{q_2} \frac{1 + i}{q_2} e^{iq_2}.
\]

(A.1)

Here, the reference to the image dipoles has disappeared, except that the distance \( q_2 \) is still there.

The electric field complex amplitude is in the \( z \) direction, so perpendicular to the surface, as it should be. For the magnetic field complex amplitude we find

\[
\mathbf{b} = -\frac{2}{q_1^2} e_z \times (q_{1,\parallel}\hat{u}_x + h_z \hat{u}_{\parallel}) \left( 1 + \frac{i}{q_1} \right) e^{iq_1} \\
+ \frac{2}{q_2^2} e_z \times (q_{1,\parallel}\hat{u}_x + h_z \hat{u}_{\parallel} + 2(h_y \hat{u}_x - h_x \hat{u}_y) e_y) \left( 1 + \frac{i}{q_2} \right) e^{iq_2},
\]

(A.2)

for a field point in the \( xy \) plane. We see that \( \mathbf{b} \) is in the \( xy \) plane, as it should be. Here we have

\[
q_{1,\parallel} = (\tilde{x}, \tilde{y} - h_y, 0), \tag{A.3}
\]

\[
\hat{u}_{\parallel} = (\hat{u}_x, \hat{u}_y, 0). \tag{A.4}
\]