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***Abstract*–The performance of reliability inference strongly depends on the modeling of the product's lifetime distribution. Many products have complex lifetime distributions whose optimal settings are not easily found. Practitioners prefer to utilize simpler lifetime distribution to facilitate the data modeling process while knowing the true distribution. Therefore, the effects of model mis-specification on the product's lifetime prediction is an interesting research area. This paper presents some results on the behavior of the relative bias (*RB*) and relative variability (*RV*) of *p*-th quantile of the accelerated lifetime (ALT) experiment when the generalized Gamma (GG_3) distribution is incorrectly specified as Lognormal or Weibull distribution. Both complete and censored ALT models are analyzed. At first, the analytical expressions for the expected log-likelihood function of the mis-specified model with respect to the true model is derived. Consequently, the best parameter for the incorrect model is obtained directly via a numerical optimization to achieve a higher accuracy model than the wrong one for the end-goal task.**

The results demonstrate that the tail quantiles are significantly overestimated (underestimated) when data is wrongly fitted by Lognormal (Weibull) distribution. Moreover, the variability of the tail quantiles is significantly enlarged when the model is incorrectly specified as Lognormal or Weibull distribution. Precisely, the effect on the tail quantiles is more significant when the sample size and censoring ratio are not large enough. Supplementary materials for this article are available online.

Keywords: Arrhenius Model, Asymptotic Bias, Asymptotic Variation, Generalized Gamma Distribution, Lognormal Distribution, Model Mis-Specification, Weibull Distribution

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1 Introduction

Manufacturers continuously strive to design and produce products with high quality and reliability in order to remain competitive in a global market. To meet this goal, nowadays the products are designed to function for a long time before they fail. However, for testing the product quality to reduce the testing time and meet the budget constraints, an accelerated life test (ALT) is utilized to evaluate the quality of data under different stress level and within a shorter time (Lawless 2011, Meeker 1984, Nelson 2009, Meeker et al. 1998).

The performance of an ALT strongly depends on choosing the model of the product's lifetime distribution. Generally, Weibull and Lognormal are two popular approaches and are among those widely used distributions in reliability engineering to fit the product's lifetime data in the literature (Somboonsavatdee et al. 2007, Pascual 2005, Lawless 2011).

Numerous studies have used Weibull and Lognormal distribution to fit the lifetime data; for instance, Basavalingappa et al. (2017) fitted the electromigration lifetime data for lower tail quantiles with Weibull and Lognormal. Pasari (2018) used the Weibull and Lognormal distributions for modeling earthquake inter-occurrence times. Li et al. (2018) implies Weibull and Lognormal distributions to reduce the stress induced by the fracturing process of brittle rocks. In the study conducted by Singh et al. (2018), the tensile strength and limit stress of the ceramic composite material are fitted by the Weibull and Lognormal distributions, respectively. Yu et al. (2018) analyzed the quality of cold in-place asphalt by fitting the air gradation and thickness with Weibull and Lognormal.

Although Weibull and Lognormal are two commonly used location-scale distributions in fitting lifetime data, they are not always the best choice for modeling lifetime data. In addition, models with two or more shape parameters are more accurate (Lin et al. 2013). The generalized Gamma (GG_3) distribution (Stacy 1962, Prentice 1974, Lawless 1980, Pham and Almhana 1995) is one of these models. GG_3 distribution with more than one shape parameter provides a flexible model to fit the reliability data. However, due to some limitation GG_3 may not be the desirable choice for the decision maker in fitting the lifetime data. The failure may appear due to 1) inadequate prior knowledge regarding the system and its dynamics, 2) relying on the availability of a large set of training examples to derive complex models, 3) preferring low accuracy model in order to forsake the computation complexity, 4) lack of closed-form expression for the maximum likelihood estimator (MLE), and 5) existence of local MLEs. In addition, Weibull and Lognormal are the special cases of GG_3 distribution and more common and easy to fit for lifetime data. Therefore, the objective of this study focuses on the effect of model mis-specification of an ALT experiment when GG_3 distribution is either mis-specified as Lognormal or Weibull distribution.

For mis-specified models, White (1982) developed a methodology for deriving the asymptotic distribution of MLEs under certain regularity conditions (e.g., consistency, asymptotic normality, and Fisher information). Later, Chow (1984) emphasized that the properties of mis-specified models are corrected if and only if data are independent and identically distributed. Bai et al. (1992) performed Monte Carlo simulations to characterize the estimation of quantiles with complete (uncensored) data under mis-specified Gamma, Weibull, and Lognormal distributions. Pascual (2005) derived expressions for the asymptotic distribution of MLEs of model parameters for the p-th quantile of mis-specified Lognormal and Weibull for censored data. Their result was extended by Pascual and Montepiedra (2005), and Pascual (2006) for the ALT experiment for type I censored data. Later, Yu (2012) extended these results for the interval of quantile. Subsequently, similar works have been done by

Yokoyama (2016, 2015) for the covariate parameter of Weibull and Lognormal under model mis-specification.

Other relevant studies hinted the approach of White (1982) are included the study of Yu (2007) and Yu (2009) for modeling the mis-specification analysis between normal and extreme value distributions for linear regression models. In the study of Peng and Tseng (2009), they investigated the mis-specification analysis of linear degradation models. Following their study, Tsai et al. (2011) applied their approach for the mis-specification analysis of Gamma and Wiener degradation processes. Rigollet et al. (2012) defined the Kullback-Leibler aggregation for measuring the distance between the true model and the wrong model for mis-specified generalized linear models for the exponential family distribution. Ling and Balakrishnan (2017) analyzed the reliability assessment of lifetime data under mis-specified Weibull and Gamma distributions. Yu and Huang (2017) investigated random-intercept mis-specification in generalized linear mixed models for binary responses.

In this study, to address the effects of model mis-specification, following the result of White (1982), first the analytical expression for expected log-likelihood function when GG_3 distribution is either mis-specified as Lognormal or Weibull distribution is derived. Then, the best parameter for the wrong model is obtained directly by using the wrong model under the expectation of GG_3. Furthermore, the relative bias (RB) and relative variability (RV) are defined to measure the accuracy and precision of the estimated p-th quantile of the product's lifetime distribution for both complete and censored ALT models.

The rest of this study is organized as follows. Section 2 utilizes some datasets appeared in literature to state the motivation of the study. Section 3 addresses the effect of model mis-specification and the result of analytical approaches. Section 4 presents a simulation study when the sample size is finite. Section 5 investigates the effects of model mis-specification on the real case study. Finally, some concluding remarks are made in Section 6. All the technical details are given in the Appendices.

2 Motivating Examples and Problem Formulation

Nowadays, the product's lifetime is highly reliable. In this case, an ALT experiment shall be conducted and apply an extrapolation for estimating the lifetime information under normal stress condition.

Reviewing the application of Weibull and Lognormal for the aforementioned cases in Section 1 clarifies how the effect of model mis-specification could be serious in practice. For instance, in the study of Basavalingappa et al. (2017), since the probability of failure for integrated circuit (IC) devices is 1 in a billion or lower, the tail quantiles are of extreme importance in lifetime analysis of IC manufacturing. Therefore, Lognormal and Weibull are performed significantly different, even with the small percentage of failure. Hereupon, conducting the accelerated test, the effect of model mis-specification under the normal stress level will cause a significant effect on the prediction of the product's lifetime.

Another notable example is addressed in the study of Pasari (2018). Due to the rare events and time-dependent behavior of high magnitude earthquakes, the effect of model mis-specification on predicting the inter-occurrence times of high magnitude earthquakes is very significant. As reported in Pasari (2018), the wrong selection of suitable distribution effects on predicting the event time with ± 50 years variation which is not negligible.

As reported by Singh et al. (2018), Weibull or Lognormal can be used to fit the strength and damage tolerance of Silicon carbide (SiC) fiber-reinforced SiC matrix composites as one of the high consistent components in a high-temperature environment. High consistency of SiC-SiC makes it difficult to observe the damage and fatigue data and predict the probability of failure of this composite in high temperature. Therefore, consider the application of SiC-SiC in a hypersensitive industry such as nuclear power plant, the effect of model mis-specification is very significant and vital for damage or fatigue prediction of SiC-SiC composite.

In this study, to investigate the effect of model mis-specification, a 3-parameter generalized gamma lifetime distribution, $GG_3(\kappa,\alpha,\beta)$, is adopted as follows:

$$f\left(t,\kappa,\alpha,\beta\right)=\begin{cases}\dfrac{\beta}{\Gamma(\kappa)\alpha}\left(\dfrac{t}{\alpha}\right)^{\kappa\beta-1} exp\left[-\left(\dfrac{t}{\alpha}\right)^{\beta}\right] & t>0\\ 0 & t<0\end{cases} \qquad (1)$$

where $\alpha > 0$, is a scale parameter, $\kappa > 0$ and $\beta > 0$ are shape parameters, and $\Gamma(\kappa)$ is the gamma function of κ,:

$$\Gamma(\kappa)=\int_0^{\infty} s^{\kappa-1}e^{-s}ds. \quad (2)$$

Note that when $\kappa = 1$, GG_3 turns to the Weibull distribution and for $\kappa \to \infty$, GG_3 tends to be the Lognormal distribution. The GG_3 covers many commonly used distributions and is particularly unable of estimating the MLE due to lack of closed-form expression and the existence of local MLEs (see Prentice (1974), Farewell and Prentice (1977)).

Therefore, the objective of this study is if the data originally comes from a GG_3 distribution, but wrongly fitted by Weibull or Lognormal, then “what is the effect of model mis-specification on the estimating the properties of product’s lifetime distribution?” To clarify this situation, consider the ball-bearing data of Lieblein and Zelen (1956) has true distribution as $GG_3(\kappa=10.22,\alpha=1.55,\beta=0.61)$. This data set can be wrongly fitted by Weibull or Lognormal distribution, due to the close performance of both distributions to GG_3, as shown in Figure 1 and 2. Therefore, a simulation study is used to address the effect of model mis-specification. For each simulation trial, a random sample of sizes $n=10,15,25,50$

is generated from GG_3 distribution and then fitted by Weibull and Lognormal distribution.

Let $\hat{t}_{pnl}(wrong)$ denotes the result of l-th simulation trial for the estimated p-th quantile, based on the wrong lifetime model under sample size n where the incorrect lifetime model is either Weibull or Lognormal distribution. Then, the empirical RB and RV for estimated p-th quantile are defined as follows, respectively:

$$K_{pn}\left(wrong\right)=\frac{\bar{t}_{pn}\left(wrong\right)-\bar{t}_{pn}\left(GG_3\right)}{\bar{t}_{pn}\left(GG_3\right)}, \qquad (3)$$

and

$$\rho_{pn}\left(wrong\right)=\frac{\sum_{l=1}^{L}\left[\hat{t}_{pnl}\left(wrong\right)-\bar{t}_{pn}\left(GG_3\right)\right]^2}{\sum_{l=1}^{L}\left[\hat{t}_{pnl}\left(GG_3\right)-\bar{t}_{pn}\left(GG_3\right)\right]^2}, \qquad (4)$$

where $\bar{t}_{pn}=\frac{1}{L}\sum_{l=1}^{L}\hat{t}_{pnl}$.

According to the L = 1000 simulation trails, the results of RB(RV) under various combinations of p-th quantile and sample size n, are shown in Table 1.

The corresponding results reveal that the effects of model mis-specification are not negligible, especially for the tail quantiles.

2.1 Assumptions

Consider in an ALT experiment, S_0 denotes the used-stress level, S_j defines the applied-stress level, S_m is the predetermined upper bound on S, and $S_0 < S_1 < \ldots < S_j < \ldots < S_m$ implicate the environments of m higher level of testing stress. Therefore, for $\left\{S_j\right\}_{j=1}^{m}$ test levels, assume that $\left\{n_j\right\}_{j=1}^{m}$ sample lifetime data are selected to perform ALT experiment with $\left\{c_j\right\}_{j=1}^{m}$ as the corresponding censoring ratio.

To define $\left\{n_j\right\}_{j=1}^{m}$, let n be the total sample size and $\pi_1,\ldots,\pi_m$ be the sample size allocation properties, where

$$\sum_{j=1}^{m}\pi_j = 1, \quad 0 \le \pi_j \le 1 \quad (5)$$

then

$$n_j = n * \pi_j.$$

In this study, the case of m = 3 is selected and ratio 4:2:1 is adopted for sample size allocating under low:middle:high stress levels, respectively.

To define the product's lifetime distribution under ALT experiment, let T_{ij} be the i-th observation (lifetime data) under applied stress level, S_j, for $1 \le i \le n_j, 1 \le j \le m$. Assume that $log(T_{ij})$ follows a Log-location-scale distribution as follows:

$$log\left(T_{ij}\right) = \mu_0\left(S_j\right) + \sigma\epsilon_{ij}, \qquad (6)$$

where μ is location parameter, σ is scale parameter, and

$$\mu_0\left(S_j\right) = \gamma_{00} + \gamma_{01}X\left(S_j\right), \qquad (7)$$

and γ_{00}, γ_{01} are unknown intercept and slope parameters of location parameter μ_0, respectively. In addition,

$$X(S_j) = \frac{1}{273.15 + S_j} \quad (8)$$

is considered as the applied Arrhenius reaction model in Balakrishnan et al. (2017) for designing the ALT experiment in this study.

The standardized stress level of S_j for $1 \le j \le m$, named x_j can be defined as:

$$x_j = \frac{X(S_0) - X(S_j)}{X(S_0) - X(S_m)}. \quad (9)$$

Note that the standardized used-level and standardized upper bound on S are $x_0 = 0$ and $x_m = 1$, respectively. Regardless the design of ALT test, equation (9) remains the same for all ALT designs that meet the $X(S_m) < X(S_j) < X(S_0)$ condition, and can be changed to

$$x_j = \frac{X(S_j) - X(S_0)}{X(S_m) - X(S_0)}. \quad (10)$$

where $X(S_0) < X(S_j) < X(S_m)$.

For other designs of ALT experiments, one can refer to Kececioglu and Jacks (1984).

Now, let ϵ_{ij} be the white noise for i-th observation under j-th stress level with the standard cumulative distribution function (CDF), $\epsilon_{ij} \sim \Phi(\cdot)$, then:

$$F_{T_{ij}}(t_{ij}) = \Phi\left(\frac{\log t_{ij} - \mu(x_j)}{\sigma}\right), \quad (11)$$

where

$$\mu(x_j) = \gamma_0 + \gamma_1 x_j, \qquad (12)$$

and γ_0 and γ_1 are the re-parameterizations of γ_{00} and γ_{01}.

Consider the $log(T_{ij})$ is the log of lifetime data with censored time $log(\eta_j)$, the equivalent log-location-scale lifetime models of GG_3, Weibull, and Lognormal are called Log Gamma, Smallest Extreme Value (SEV), and Normal distribution, respectively. The corresponding models are denoted by M_{LG}, $M_{SEV,}$ and M_{Nor}, respectively as follows:

$$M_{LG} : \log(T_{ij}) \sim \mathrm{LG}\left(\kappa, \mu_g\left(x_j\right) = \gamma_{0g} + \gamma_{1g} x_j, \sigma_g\right), \qquad (13)$$

$$M_{SEV} : \log(T_{ij}) \sim \mathrm{SEV}\left(\mu_e\left(x_j\right) = \gamma_{0e} + \gamma_{1e} x_j, \sigma_e\right), \qquad (14)$$

and

$$M_{Nor} : \log(T_{ij}) \sim \mathrm{Nor}\left(\mu_N\left(x_j\right) = \gamma_{0N} + \gamma_{1N} x_j, \sigma_N\right). \qquad (15)$$

where g, e, and N are subscripts for parameters of Log Gamma, SEV, and Normal distribution, respectively.

For log-location-scale distribution, the p-th quantile of lifetime data is denoted by $t_p(x_0)$ under the standardized stress x_0 as follow:

$$t_p(x_0) = \exp\left(\mu(x_0) + \sigma\Phi^{-1}(p)\right) \qquad (16)$$

Therefore, the probability of observing failure at standardized stress x_j by censoring time $log(\eta_j)$ can be described by:

$$p_j = \Phi\left(\frac{\log \eta_j - \mu(x_j)}{\sigma}\right). \qquad (17)$$

3 The Effects of Model Mis-specification

As it is mentioned earlier, this study aims to define the RB and RV of the estimated p-th quantile of the product's lifetime distribution for both complete and censored ALT models to measure the accuracy and precision when the true model is GG_3 but wrongly fitted by Weibull or Lognormal. Following steps identify the procedure of study the effects of model mis-specification on *RB* and *RV* for ALT complete and censored data:

Step 1: Find the asymptotic distribution and best parameter setting of the wrong distribution with respect to the true model (Section 3.1).

Step 2: Derive the *RB* and *RV* of a function of a random variable (e.g., *p*-th quantile in this study) under ALT complete and censored data (Section 3.2, and 3.3).

3.1 Asymptotic Distribution of Estimators

In this section, the results of White (1982) are used to derive the asymptotic distribution of the MLEs for referring to the properties of the underlying mis-specified model. These incorrect MLEs are called quasi-MLEs (QMLEs). In the following, the vector of MLEs and QMLEs under the model M_k are shown as $\boldsymbol{\theta}_k$ and $\hat{\boldsymbol{\theta}}_k$, respectively.

Let $k(k')$ be the subscript when the correct (fitted) model is used, where $k \neq k'$. Let M_k and $M_{k'}$ be the correct and fitted models, respectively. Assume $\Im_k(\boldsymbol{\theta}_k)$ and $\Im_{k'}(\boldsymbol{\theta}_{k'})$ be log-likelihood functions under the model M_k and $M_{k'}$, respectively. The Kullback-Leibler distance (Joyce 2011) utilizes the expected value with respect to the true model (E_{M_k}) to measure the distance between the correct and fitted models as follow:

$$I(\boldsymbol{\theta}_{\mathbf{k}}, \boldsymbol{\theta}_{k'}) = E_{M_k}\left(\Im_k(\boldsymbol{\theta}_k) - \Im_{k'}(\boldsymbol{\theta}_{k'})\right) \quad (18)$$

For fixed $\boldsymbol{\theta}_k$, let $\boldsymbol{\theta}^*_{k'}$ be the value of $\boldsymbol{\theta}_{k'}$ that minimize the expected negative likelihood $E_{M_k}\left(-\Im(\boldsymbol{\theta}_{k'})\right)$ with respect to M_k:

$$\boldsymbol{\theta}^*_{k'} = \operatorname{argmin}_{\boldsymbol{\theta}_{k'}}\left[E_{M_k}\left(-\Im_{k'}(\boldsymbol{\theta}_{k'})\right)\right]. \quad (19)$$

where $\boldsymbol{\theta}_{k'}^{*}$ is called asymptotic value of $\boldsymbol{\theta}_{k'}$, and the best parameter setting under $M_{k'}$ with respect to the true model. Therefore, when the true model comes from M_k, by theorem 3.2 in White (1982), and $\delta-$method (Oehlert 1992), the asymptotic distribution of $\boldsymbol{\theta}_{k'}$ as the QMLE of $\boldsymbol{\theta}_{k'}$ is:

$$\sqrt{n}\left(\boldsymbol{\theta}_{k'}-\boldsymbol{\theta}_{k'}^{*}\right)\xrightarrow{d} Normal\left(0, C\left(\boldsymbol{\theta}_{k}, \boldsymbol{\theta}_{k'}=\boldsymbol{\theta}_{k'}^{*}\right)\right), \qquad (20)$$

where

$$C(\boldsymbol{\theta}_k,\boldsymbol{\theta}_{k'})=\left[A(\boldsymbol{\theta}_k,\boldsymbol{\theta}_{k'})\right]^{-1}B(\boldsymbol{\theta}_k,\boldsymbol{\theta}_{k'})\left[A(\boldsymbol{\theta}_k,\boldsymbol{\theta}_{k'})\right]^{-1}, \qquad (21)$$

is the variance-covariance matrix of QMLE, and

$$A(\boldsymbol{\theta}_k,\boldsymbol{\theta}_{k'})=\left[E_{M_k}\left(\frac{\partial^2\Im_{k'}(\boldsymbol{\theta}_{k'})}{\partial\theta_{k'r}\partial\theta_{k's}}\right)\right] \qquad (22)$$

and

$$B(\boldsymbol{\theta}_k,\boldsymbol{\theta}_{k'})=\left[E_{M_k}\left(\frac{\partial\Im_{k'}(\boldsymbol{\theta}_{k'})}{\partial\theta_{k'r}}\frac{\partial\Im_{k'}(\boldsymbol{\theta}_{k'})}{\partial\theta_{k's}}\right)\right], \qquad (23)$$

are expected values of partial derivatives of the log-likelihood function of the correct model with respect to the fitted model, where $\theta_{k'r}$ is the *r*-th element of $\boldsymbol{\theta}_{k'}$.

The elements of matrices *A* and *B* when the true distribution is GG_3 and mistreated by Weibull or Lognormal is derived in Appendix A (see supplementary materials)

3.2 *RB* and *RV* of Function of QMLE

For a given function *g*, let $g(\boldsymbol{\theta}_{k'})$ be the QMLE of $g(\boldsymbol{\theta}_{k})$, then the asymptotic bias term for $g(\boldsymbol{\theta}_{k'})$ is defined as:

$$\mathrm{ABias}[g(\boldsymbol{\theta}_{k'}) \mid M_k] = E_{M_k}(g(\boldsymbol{\theta}_{k'})) - g(\boldsymbol{\theta}_k) = g(\boldsymbol{\theta}^*_{k'}) - g(\boldsymbol{\theta}_k). \quad (24)$$

Therefore, the *RB* of *g* under the model mis-specification is:

$$RB = \frac{\mathrm{ABias}[g(\boldsymbol{\theta}_{k'}) \mid M_k]}{g(\boldsymbol{\theta}_k)}. \quad (25)$$

Similarly, let $Avar[g(\boldsymbol{\theta}_{k'}) \mid M_k]$ be the asymptotic variance and $AMSE(g(\boldsymbol{\theta}_{k'}) \mid M_k)$ be the asymptotic mean square error of $g(\boldsymbol{\theta}_{k'})$ where M_k is the true model. Then,

$$AMSE(g(\boldsymbol{\theta}_{k'}) \mid M_k) = Avar[g(\boldsymbol{\theta}_{k'}) \mid M_k] + (\mathrm{ABias}[g(\boldsymbol{\theta}_{k'}) \mid M_k])^2. \quad (26)$$

Therefore, the *RV* of *g* under the model mis-specification is:

$$RV = \frac{AMSE(g(\boldsymbol{\theta}_{k'}) \mid M_k)}{AMSE(g(\boldsymbol{\theta}_k) \mid M_k)}, \quad (27)$$

where $AMSE(g(\boldsymbol{\theta}_k) \mid M_k) = var[g(\boldsymbol{\theta}_k)]$.

3.3 *RB* and *RV* of *p*-th Quantile under ALT Experiment

Consider the function *g* as the *p*-th quantile of a lifetime distribution which satisfying the notations and assumptions in Section 3.2.

Let Z_{ijk} be the standardized log lifetime, and ζ_{ijk} be the standardized log censoring time under the model M_k for the *i*-th sample of *j*-th stress level where $1 < i < n_j$, and $1 < j < m$, calculated as:

$$Z_{ijk} = \frac{\log T_{ij} - \mu_k(x_j)}{\sigma_k} \quad (28)$$

and

$$\zeta_{ijk} = \frac{\log \eta_j - \mu_k(x_j)}{\sigma_k}. \quad (29)$$

Let $\phi(\cdot)$ and $\Phi(\cdot)$ be the standard probability density function (PDF) and CDF under M_k, respectively. Then, the corresponding log-likelihood function for model M_k is:

$$\Im_k = \sum_{j=1}^{m}\left\{\sum_{i=1}^{n_j}\left\{\delta_i \log\left(\frac{1}{\sigma_k}\phi_k(z_{ijk})\right) + (1-\delta_i)\log\left(1-\Phi(\zeta_{jk})\right)\right\}\right\}, \qquad (30)$$

where

$$\delta_i = \begin{cases} 1 \quad \log(T_{ij}) \le \log(\eta_j), \quad \text{if} \quad \log(T_{ij}) \quad \text{is} \quad \text{observed} \\ 0 \quad \log(T_{ij}) \le \log(\eta_j), \quad \text{if} \quad \log(T_{ij}) \quad \text{is} \quad \text{censored} \end{cases} \qquad (31)$$

Applying the models in (13), (14) and (15), the corresponding log-likelihood function, the expected negative log-likelihood, the expected value where $\eta_j \to \infty$ and the *p*-th quantile of the incorrect model with respect to the true model under the ALT test are summarized in Table 2.

For the $g(\boldsymbol{\theta_k}) = t_p(\boldsymbol{\theta_k})$, Equation (25) turns to:

$$RB_{k'} = \frac{t_p(\boldsymbol{\theta}_{k'}^*) - t_p(\boldsymbol{\theta}_k)}{t_p(\boldsymbol{\theta}_k)}. \qquad (32)$$

The derivation of t_p with respect to $\boldsymbol{\theta}_k = [\gamma_0, \gamma_1, \sigma]$ is:

$$\frac{\partial t_p(\boldsymbol{\theta}_k)}{\partial \boldsymbol{\theta}_k} = \left[1, x_j, \Phi_k^{-1}\right]' t_p(\boldsymbol{\theta}_k), \quad (33)$$

and the asymptotic variance is:

$$Avar(t_p(\boldsymbol{\theta}_{k'}) \mid M_k) = \frac{1}{n} t_p(\boldsymbol{\theta}_{k'})\left[1, x_j, \Phi_{k'}^{-1}(p)\right] C\left(\boldsymbol{\theta}_k; \boldsymbol{\theta}_{k'} = \boldsymbol{\theta}_{k'}^*\right)\left[1, x_j, \Phi_{k'}^{-1}(p)\right]' t_p(\boldsymbol{\theta}_{k'}). \ (34)$$

Therefore, Equation (27) reduces to:

$$RV_{k'} = \frac{Avar(t_p(\boldsymbol{\theta}_{k'}) \mid M_k) + \left[t_p(\boldsymbol{\theta}^*_{k'}) - t_p(\boldsymbol{\theta}_k)\right]^2}{var(t_p(\boldsymbol{\theta}_k))} \qquad (35)$$

To illustrate how the theoretical approach can be computed from sample data, consider a 3-level ALT experiment with the standardized stress levels at low, medium and high rates as $(x_L, x_M, x_H) = (0.25, 0.5, 1)$.

Assume that the underlying lifetime distribution follows a GG_3 with $\alpha(x_j) = 1.55 - 0.8x_j$, $\beta = 0.61$ and $\kappa = 10.22$. Now, if the Weibull and Lognormal are wrongly used to fit this ALT model with $(\pi_1, \pi_2, \pi_3) = (4/7, 2/7, 1/7)$, then their corresponding best parameter-setting can be obtained directly via minimizing the expected negative log-likelihood where $\eta_j \to \infty$ (P_3 in Table 2). The results are shown in Table 3.

Figure 3 and 4 show the theoretical results of *RB* and *RV* for *p*-th quantile under various settings of *p* for Weibull and Lognormal distributions, respectively. Comparing the quantiles, if the true distribution is GG_3 and the fitted distribution is incorrect, there could be a significant amount of bias and variation in estimating tail quantiles. The results from Figure 3 show that the *RB* of tail quantiles are negative (underestimate) for Weibull distribution and are positive (overestimate) for Lognormal distribution. In addition, the effect of model mis-specification is more significant for Weibull distribution than the Lognormal distribution in this example.

The shape parameter, κ, of GG_3 plays an important role in model mis-specification. Derived by this fact the behavior of Lognormal distribution was more close to GG_3 than Weibull in Figures 3 and 4 due to the setting of $\kappa = 10.22$. Therefore, to investigate the effect of shape parameter on estimating the accuracy and precision of the product's *p*-th quantile, the result of the best parameter setting under various combinations of the parameter κ is verified. The result of *ABias* and *AMSE* are depicted in Figures 5 and 6, respectively. As it was expected when $\kappa = 1$, both *ABias*, and *AMSE* of Weibull distribution are equal to zero. The results show that by increasing the value of κ, both *ABias* and *AMSE* of Weibull are increasing while Lognormal distribution has the decreasing pattern, and apparently for $\kappa \geq 10$, Lognormal has a better performance than the Weibull distribution.

4 Simulation Study

4.1 The Case of Complete Data

The results in Section 3 are based on the infinite sample property of model mis-specification. In practical application, the sample size cannot be infinite. Therefore, the Monte Carlo simulation experiment is utilized to investigate the effect of the sample size on estimating the penalty of choosing incorrect models. It is expected that by increasing the sample size, QMLE's results convergence to the theoretical result in Section 3.2. Similar to (3), for the simulated data the empirical *RB* for *p*-th quantile is:

$$K_p = \frac{\bar{t}_p(\boldsymbol{\theta}_{k'}) - \bar{t}_p(\boldsymbol{\theta}_g)}{\bar{t}_p(\boldsymbol{\theta}_g)} \quad (36)$$

where regardless the distribution $\bar{t}_p = \frac{1}{L}\sum_{l=1}^{L} t_{lp}$, and t_{lp} denotes the *p*-th quantile for the *l*-th observation of *L* simulation trails. Subsequently equivalent to (4), the empirical *RV* is:

$$\rho_p = \frac{\sum_{l=1}^{L}\left[t_{lp}(\boldsymbol{\theta}_{k'}) - \bar{t}_p(\boldsymbol{\theta}_g)\right]^2}{\sum_{l=1}^{L}\left[t_{lp}(\boldsymbol{\theta}_g) - \bar{t}_p(\boldsymbol{\theta}_g)\right]^2}. \quad (37)$$

The Monte Carlo simulation is investigated to see how effectively the asymptotic behavior matches the sample size behavior of the observed bias and variance. For this purpose, "flexsur", an R package for fully-parametric modeling of survival data is employed to fit the lifetime distribution which offers two different versions of GG_3 distribution, "gengamma.orig" for an original parameterization in (1) and " gengamma" for the stable parameterization form in Prentice (1974).

The data simulation steps are as follow:

1. Generate $n = (35, 70, 140, 280)$ data set based on the $GG_3(\kappa = 10.22, \alpha(x_j) = 1.55 - 0.8x_j, \beta = 0.61)$ and for each stress level under test plan 4:2:1, $(x_L, x_M, x_H) = (0.25, 0.5, 1)$
2. Fit models M_{LG}, $M_{SEV,}$ and M_{Nor} to the new data exclusively.
3. Compute the p-th quantile (P_4 in Table 2) where parameters are estimated from P_3 in Table 2, for each model under the applied-stress level, x_j, for $p = 1, ..., 100$.
4. Replicate steps 1-3 for L = 1000 times.
5. Estimate RB in (36) and RV in (37).
6. Iterate steps 1-5 for different sample size according to Table 4.

The simulation results of observed RB and RV for complete data are given in Table 5. The result is compared with the theoretical outcome. It was expected that simulation results converge to theoretical ones for large sample size, which is observable in Table 5. The pairwise plots of RB and RV for Weibull and Lognormal over all quantiles for the case of complete data are shown in Appendix C (see the supplementary materials). From the results it can be seen that the maximum RV of Lognormal is extremely smaller than the Weibull due to the value of $\kappa \approx 10$ which validates the convergence of Lognormal to GG_3.

The results show that if the distribution is mis-specified, there could be a significant amount of bias in estimating the tail quantiles. The outcome of the simulation experiment is consistent with analytical results from Figure 3 and 4. In addition, RB and RV have a similar pattern in which only the lower tail quantiles are significantly enlarged for Weibull distribution. Although in comparison with

Weibull distribution the effect of *RB* and *RV* for Lognormal distribution is negligible, bias and variation are increased for both tails of Lognormal distribution. This behavior was predictable from comparing density plots in Figure 1. The results answer how large sample size is sufficient to have an in-control bias and variance from the mis-specified model. Typically, the effect of sample size is entirely negligible for *RB*, and Lognormal is a more appropriate model than the Weibull in this experiment due to the value of κ. However, the *RV* significantly departs from 1 when the sample size becomes larger.

4.2 The Case of Censoring Data

Consider Type I censored data (Chandra and Khan 2013) in order to address the effect of the censoring ratio on the model mis-specification under ALT experiment. Assume that η_j is the censoring time under the *j*-th stress level for j = 1, 2, 3 indicating low, medium, and high stress levels. The censoring ratio is defined as follow:

$$c_j = \frac{\eta_j}{\mathrm{MTTF}_j} \quad (38)$$

where MTTF_j is the product's mean lifetime to failure under stress level *j*.

The data simulation steps for censored data are as follow:

1. Use the simulated uncensored data in Section 4.1.
2. Choose the censoring ratio $c_j = 2, 1.6, 1.2$ for j = 1, 2, 3 to create the censored data, and repeat the following steps for each stress level exclusively.
3. Fit model M_{LG}, $M_{SEV,}$ and M_{Nor} to the new censored data exclusively.
4. Compute the *p*-th quantile (P_4 in Table 2) where parameters are estimated from P_2 in Table 2, for $p = 1, \ldots, 100.$
5. Replicate steps 1-4 for L = 1000 times.
6. Estimate *RB* in (36) and *RV* in (37).

7. Iterate steps 1-6 for different sample size according to Table 4.

The pairwise plots of *RB* and *RV* for Weibull and Lognormal over all quantiles and different sample sizes for the case of censored data are shown in Figures 7 and 8, respectively.

Figure 7 demonstrates that *RB* is reduced by increasing the censored points for Weibull distribution. In general, for a model with lower censoring ratio by decreasing sample size *n*, Weibull can fit the data better than the Lognormal. This is due to the nature of Weibull and Lognormal distributions. Weibull can present the skewness of log-data, while Lognormal is symmetrical for log-data. Therefore, with more censoring points data is skewness to the left and Weibull is performed better than Lognormal especially in the upper tail quantiles. The increasing pattern of *RB* and *RV* by the growth in censoring ratio for Lognormal can verify this phenomenon. In addition, as it is demonstrated in Figure 2 Lognormal is unsteady in estimating the upper tail quantiles. This behavior is due to the advantages of Weibull distribution for having lighter tails and therefore smaller kurtosis in comparison with the log-normal distribution (Rousu 1973). Other notable findings from Figure 7 are as follows:

Data sets with low kurtosis tend to have light tails

1. Except for a few cases, the *RB* for Lognormal (Weibull) case is positive (negative). It means that the *p*-th quantile of Lognormal (Weibull) is over-estimated (under-estimated).
2. In general, the absolute *RB* for the Weibull distribution is slightly smaller than Lognormal distribution.
3. Regardless of the sample size, for all cases, the *RB* of Weibull for tail quantiles decreases as censoring ratio decreases. For the case of Lognormal, only the *RB* of upper tail quantiles decreases as the censoring ratio increases.

4. Regardless of the value of c_j, for all cases, the *RB* of lower tail quantiles of Lognormal distribution stays unchanged. Furthermore, under the same sample size, the *RB* of upper tail quantiles for Lognormal distribution is more significant than Weibull, more obviously for the smaller value of c_j.

Subsequently, Figure 8 represents the result of *RV* and supports the outcomes of *RB* in Figure 7. Figure 8 shows how the variation for tail quantiles is changing by reducing c_j, such that the higher quantiles have more significant variation than the lower ones, more specifically for Lognormal distribution. This is in contrast with the result of complete data in Section 4.1, such that for complete data the variation at lower tails is more significant than the higher ones. General findings from Figure 8 are as follows:

1. For the case of Lognormal (Weibull) distribution, the *RV* of lower and upper tail quantiles increases (decreases) when the censoring ratio decreases.
2. For the case of Lognormal, the *RV* of the upper tail quantiles is increased significantly by decreasing the censoring ratio.
3. For both Weibull and Lognormal distributions, the *RV* of lower and upper tail quantiles decreases when the sample size increases. While for the case of complete data, *RV* of lower tail quantiles increases when the sample size increases. This is due to the increase in “between distribution variation” and the decrease in “within distribution variation” by the growth of sample size for complete data. However, increases in sample size for censored data is induced higher similarity among data and therefore lower variation for both within and between distribution situations.

General speaking, the simulation study shows that the effects of model mis-specification for the ALT experiment of censored data on the tail quantiles are significant when the sample size and censoring ratio are not large enough.

5 Beyond the Effects of Model Mis-specification on Product Lifetime

In Section 4, through the simulation study, it is shown how critical is the effects of model mis-specification on product lifetime, especially for the tail quantiles. However, in the larger stage of the product life cycle, the effect of model mis-specification would be more critical if the lifetime analysis in operation stage did not estimate correctly. Some of these stages are 1) product service and warranty period estimation, 2) environmental testing under the customers' use condition, 3) inventory management, shelf-life extension, and batch trading, 4) maintenance planning/condition-based maintenance, and 5) damage analysis and fatigue testing.

For demonstrating the broader effects of model mis-specification in real applications, warranty planning is selected in this section for further investigation.

Warranty is an important sales feature for many products. Manufacturers usually want to predict the warranty cost as early as possible for various purposes. This can be controlled by using the ALT data from the early stages of the product life cycle (McSorley et al. 2002). When data from the early stages of the product life cycle is used to establish warranties and service policies, the interests of the producer and consumer should be balanced. From the standpoint of safety, long projected service life could harm the consumer. Establishing a short warranty period hurts revenue. To estimate an affordable warranty length, manufacturers often use ALT test for designing different warranty lengths under the customers' use condition (Yang 2010).

In addition, increasing use-rate during the warranty period is an effective index on ALT data. To illustrate this situation, the scenario for simulating warranty data in Blischke et al. (2011) is considered, such that the standardized use-rate for (x_L, x_M, x_H) sets as $(0, 0.5, 1)$, with corresponding (7, 14, 56) sample size of failure data and a total of 100 records at each rate. In addition, it is considered that data are originally come from $GG_3(\alpha = 9.9 - x_j, \beta = 1, \kappa = 1)$ distribution. Regarding the value of κ = 1, Weibull distribution as the most comparable distribution is considered for fitting the data. Figure 9 illustrates the *RB* and *RV* for the case when Weibull distribution is selected to fit the data while GG_3 is the true distribution for 365 days of simulated data. The main finding in Figure 9 shows that even when $\kappa = 1$, and Weibull tends to GG_3 distribution, still there is a significant effect of model mis-specification on *RB* and *RV*. The main issue to fit the GG_3 data when $\kappa = 1$ with Weibull distribution is that there is a slight bias with respect to κ which only large sample size can decline it (for more information one can refer to Hager et al. (1971)).

Table 6 summarized the MTTF for each use-rate level under the Weibull and GG_3 distributions. The result shows that there is almost 10% bias in estimating the product lifetime when Weibull distribution wrongly fits the data. This situation when it comes under the warranty policy, estimating warranty length and cost estimation would be even more tangible and critical for decision makers.

6 Conclusion

This paper investigated the impact of model mis-specification of ALT plan under the GG_3 distribution for both censored and complete data. The asymptotic distribution of p-th quantile of the product's lifetime under mis-specified distribution is derived. Furthermore, a simulation study based on empirical data is carried out to evaluate the penalty of the model mis-specification. From the results, it is observed that:

1. The *RB* of tail quantiles are significantly overestimated (underestimated) when GG_3 is wrongly fitted by Lognormal (Weibull) distribution, and the variability of corresponding tail quantiles is significantly enlarged.
2. The prediction of a product's p-th quantile for the large enough sample size converges to the theoretical result.
3. The effect of sample size could be negligible on *RB* for both complete and censored data, and both Weibull and Lognormal distributions.

4. The effect of sample size could be negligible on *RV* for the case of Lognormal distribution with complete data.
5. By decreasing the censoring ratio, Weibull is a better model than Lognormal for fitting the GG_3's data due to more skewness of data to the left. This result is validated by Kim and Yum (2008) for moderate to heavily censored cases if the sample size exceeds a certain threshold. In general, when data has skewness Weibull distribution is a preferred choice.
6. The *RV* for complete data increases when the sample size increases, while for censored data decreases when the sample size increases.
7. Consequently, the *RV* of lower quantiles is significantly enlarged for complete data, more specifically for Weibull distribution. However, the *RV* of higher quantiles is significantly enlarged for censored data, more specifically for Lognormal distribution.

At the end of this section, the following issues are worthwhile for future research:

- In section 3, for different values of κ, due to the complexity involved in estimating the θ^* under censored data, only the mis-specification problem under the case of complete data is addressed. Obviously, under the type-I censoring scheme, study the effect of the parameter κ on model mis-specification shall be a challenging issue for future research.
- In practical applications, there are situations that the true distribution comes from a mixture distribution. However, it may wrongly fit by a unimodal lifetime distribution. Therefore, how significant is the effect of the model mis-specification when the true distribution is mixture distribution but is wrongly fitted by unimodal distribution, should be an interesting research topic.
- Since Weibull (Lognormal) distribution often underestimates (overestimates) the tail quantiles, investigating on a model that combine these two distributions to get a more accurate quantile estimation it is

worth. Therefore, a mixture of Weibull-Lognormal distribution that can explain both multiplicative effect behavior of Brownian motion through Lognormal distribution and additive effect behavior of non-homogeneous Poisson process through Weibull distribution could be an interesting topic to be studied as two-component distribution for modeling lifetime data.

- When the sample size is large enough the bias is not a concern for practitioners since estimators are approximately unbiased. In this situation, when the fitted model turns out to be incorrect, finding a robust test plan that reduces the bias and inefficiency of model mis-specification is necessary.

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SUPPLEMENTARY MATERIAL

Appendix A Asymptotic Variance-Covariance Matrix of Mistreated Distributions
Appendix B The Derivation of P_2 and P_3 in Table 2.
Appendix C The pairwise plots of *RB* and *RV* for Weibull and Lognormal over all quantiles for the case of complete data.
R-Code Demo codes in R language are provided for estimating θ^*, *RB* and *RV* for theoretical result, and estimating QMLEs for both complete and censored data under ALT experiment. (supplementary-materials.R)

References

Bai, J., Jakeman, A. J. and McAleer, M. (1992), 'Estimating the percentiles of some misspecified non-nested distributions', *Journal of Statistical Computation and Simulation* **42**(3-4), 151–159.

Balakrishnan, N., Tsai, C.-C. and Lin, C.-T. (2017), Gamma degradation models: Inference and optimal design, *in* 'Statistical Modeling for Degradation Data', Springer, pp. 171–191.

Basavalingappa, A., Passage, J. M., Shen, M. Y. and Lloyd, J. (2017), Electromigration: Lognormal versus Weibull distribution, *in* 'Integrated Reliability Workshop (IIRW), 2017 IEEE International', IEEE, pp. 1–4.

Blischke, W. R., Karim, M. R. and Murthy, D. P. (2011), *Warranty data collection and analysis*, Springer Science & Business Media.

Chandra, N. and Khan, M. A. (2013), 'Optimum plan for step-stress accelerated life testing model under type-I censored samples', *Journal of Modern Mathematics and Statistics* **7**(5), 58–62.

Chow, G. C. (1984), 'Maximum-likelihood estimation of misspecified models', *Economic Modelling* **1**(2), 134–138.

Farewell, V. T. and Prentice, R. L. (1977), 'A study of distributional shape in life testing', *Technometrics* **19**(1), 69–75.

Hager, H. W., Bain, L. J. and Antle, C. E. (1971), 'Reliability estimation for the generalized gamma distribution and robustness of the weibull model', *Technometrics* **13**(3), 547–557.

Joyce, J. M. (2011), Kullback-leibler divergence, *in* 'International Encyclopedia of Statistical Science', Springer, pp. 720–722.

Kececioglu, D. and Jacks, J. A. (1984), 'The arrhenius, eyring, inverse power law and combination models in accelerated life testing', *Reliability Engineering* **8**(1), 1–9.

Kim, J. S. and Yum, B.-J. (2008), ‘Selection between Weibull and lognormal distributions: A comparative simulation study’, *Computational Statistics & Data Analysis* **53**(2), 477–485.

Lawless, J. F. (1980), ‘Inference in the generalized gamma and log gamma distributions’, *Technometrics* **22**(3), 409–419.

Lawless, J. F. (2011), *Statistical models and methods for lifetime data*, Vol. 362, John Wiley & Sons.

Li, X., Huai, Z., Konietzky, H., Li, X. and Wang, Y. (2018), ‘A numerical study of brittle failure in rocks with distinct microcrack characteristics’, *International Journal of Rock Mechanics and Mining Sciences* **106**, 289–299.

Lieblein, J. and Zelen, M. (1956), ‘Statistical investigation of the fatigue life of deep-groove ball bearings’, *Journal of Research of the National Bureau of Standards* **57**(5), 273–316.

Lin, C.-T., Chou, C.-C. and Balakrishnan, N. (2013), ‘Planning step-stress test plans under type-I censoring for the log-location-scale case’, *Journal of Statistical Computation and Simulation* **83**(10), 1852–1867.

Ling, M. H. and Balakrishnan, N. (2017), ‘Model mis-specification analyses of Weibull and gamma models based on one-shot device test data’, *IEEE Transactions on Reliability* **66**(3), 641–650.

McSorley, E. O., Lu, J.-C. and Li, C.-S. (2002), ‘Performance of parameter-estimates in step-stress accelerated life-tests with various sample-sizes’, *IEEE Transactions on Reliability* **51**(3), 271–277.

Meeker, W. Q. (1984), ‘A comparison of accelerated life test plans for Weibull and lognormal distributions and type I censoring’, *Technometrics* **26**(2), 157–171.

Meeker, W. Q., Escobar, L. A. and Lu, C. J. (1998), ‘Accelerated degradation tests: modeling and analysis’, *Technometrics* **40**(2), 89–99.

Nelson, W. B. (2009), *Accelerated testing: statistical models, test plans, and data analysis*, Vol. 344, John Wiley & Sons.

Oehlert, G. W. (1992), ‘A note on the delta method’, *The American Statistician* **46**(1), 27–29.

Pasari, S. (2018), ‘Stochastic modelling of earthquake interoccurrence times in northwest Himalaya and adjoining regions’, *Geomatics, Natural Hazards and Risk* **9**(1), 568–588.

Pascual, F. G. (2005), ‘Maximum likelihood estimation under misspecified lognormal and Weibull distributions’, *Communications in Statistics-Simulation and Computation* **34**(3), 503–524.

Pascual, F. G. (2006), ‘Accelerated life test plans robust to misspecification of the stresslife relationship’, *Technometrics* **48**(1), 11–25.

Pascual, F. G. and Montepiedra, G. (2005), ‘Lognormal and Weibull accelerated life test plans under distribution misspecification’, *IEEE Transactions on Reliability* **54**(1), 43–52.

Peng, C.-Y. and Tseng, S.-T. (2009), ‘Mis-specification analysis of linear degradation models’, *IEEE Transactions on Reliability* **58**(3), 444–455.

Pham, T. and Almhana, J. (1995), ‘The generalized gamma distribution: its hazard rate and stress-strength model’, *IEEE Transactions on Reliability* **44**(3), 392–397.

Prentice, R. L. (1974), ‘A log gamma model and its maximum likelihood estimation’, *Biometrika* **61**(3), 539–544.

Rigollet, P. et al. (2012), ‘Kullback–leibler aggregation and misspecified generalized linear models’, *The Annals of Statistics* **40**(2), 639–665.

Rousu, D. N. (1973), ‘Weibull skewness and kurtosis as a function of the shape parameter’, *Technometrics* **15**(4), 927–930.

Singh, G., Gonczy, S., Deck, C., Lara-Curzio, E. and Katoh, Y. (2018), ‘ Interlaboratory round robin study on axial tensile properties of SiC-SiC CMC tubular test specimens’, *International Journal of Applied Ceramic Technology* **15**(6), 1334–1349.

Somboonsavatdee, A., Nair, V. N. and Sen, A. (2007), ‘Graphical estimators from probability plots with right-censored data’, *Technometrics* **49**(4), 420–429.

Stacy, E. W. (1962), ‘A generalization of the gamma distribution’, *The Annals of Mathematical Statistics* **33**(3), 1187–1192.

Tsai, C.-C., Tseng, S.-T. and Balakrishnan, N. (2011), ‘Mis-specification analyses of gamma and Wiener degradation processes’, *Journal of Statistical Planning and Inference* **141**(12), 3725–3735.

White, H. (1982), ‘Maximum likelihood estimation of misspecified models’, *Econometrica: Journal of the Econometric Society* **50**(1), 1–25.

Yang, G. (2010), ‘Accelerated life test plans for predicting warranty cost’, *IEEE Transactions on Reliability* **59**(4), 628–634.

Yokoyama, M. (2015), A study on estimation of lifetime distribution with covariates under misspecification, *in* ‘Proceedings of the World Congress on Engineering and Computer Science’, Vol. 2.

Yokoyama, M. (2016), ‘A study on estimation of lifetime distribution with covariates under misspecification for baseline distribution.’, *Engineering Letters* **24**(2), 195–201.

Yu, B., Gu, X., Ni, F. and Gao, L. (2018), ‘Microstructure characterization of cold in-place recycled asphalt mixtures by x-ray computed tomography’, *Construction and Building Materials* **171**, 969–976.

Yu, H.-F. (2007), ‘Mis-specification analysis between normal and extreme value distributions for a linear regression model’, *Communications in Statistics–Theory and Methods* **36**(3), 499–521.

Yu, H.-F. (2009), ‘Mis-specification analysis between normal and extreme value distributions for a screening experiment’, *Computers & Industrial Engineering* **56**(4), 1657–1667.

Yu, H.-F. (2012), ‘The effect of mis-specification between the lognormal and Weibull distributions on the interval estimation of a quantile for complete data’, *Communications in Statistics–Theory and Methods* **41**(9), 1617–1635.

Yu, S. and Huang, X. (2017), ‘Random-intercept misspecification in generalized linear mixed models for binary responses’, *Statistical Methods & Applications* **26**(3), 333–359.

Fig. 1 Histogram and density plot of ball-bearing data when is fitted by GG_3, Weibull, and Lognormal distributions.

Fig. 2 Quantile plot of ball-bearing data fitted by Weibull (red/dash), and Lognormal (black/solid) distributions.

Fig. 3 The estimated *RB* from theoretical result.

Fig. 4 The estimated *RV* from theoretical result.

Fig. 5 ABias of Lognormal (black/dash) and Weibull (red/solid) under different values of κ.

Fig. 6 AMSE of Lognormal (black/dash) and Weibull (red/solid) under different values of κ.

Fig. 7 *RB* of Lognormal (black/dash) and Weibull (red/solid), under different settings of censoring ratio and sample size, where X-axis indicates the p and Y-axis the *RB*.

Fig. 8 *RV* of Lognormal (black/dash) and Weibull (red/solid), under different settings of censoring ratio and sample size, where X-axis indicates to the p and Y-axis to the *RB*.

Fig. 9 *RB* and *RV* of Weibull distribution for simulated warranty data.

Table 1 *RB* (*RV*) of L = 1000 simulated data generated from $GG_3(\kappa = 10.22, \alpha = 1.55, \beta = 0.61)$.

Sample size	Distribution	Quantile				
		0.05	0.1	0.5	0.9	0.95
10	Weibull	-0.243(3.16)	-0.144(1.76)	0.033(1.36)	-0.018(1.2)	-0.053(1.22)
	Lognormal	0.056(1.27)	0.023(1.11)	-0.017(1.01)	0.03(1.14)	0.059(1.3)
15	Weibull	-0.239(3.77)	-0.139(1.96)	0.038(1.41)	-0.013(1.22)	-0.049(1.25)
	Lognormal	0.051(1.26)	0.0199(1.08)	-0.02(0.996)	0.027(1.11)	0.055(1.29)
25	Weibull	-0.229(6.15)	-0.128(2.66)	0.052(1.69)	0.0004(1.28)	-0.036(1.32)
	Lognormal	0.047(1.38)	0.016(1.11)	-0.024(1.06)	0.023(1.15)	0.051(1.43)
50	Weibull	-0.228(10.79)	-0.127(4.05)	0.054(1.9)	0.0017(1.23)	-0.038(1.37)
	Lognormal	0.0466(1.59)	0.0156(1.15)	-0.025(1.14)	0.0226(1.22)	0.0508(1.68)

Table 2 Properties of mis-specified distribution, $Q^{-1}(\kappa, p)$ is the inverse of a regularized incomplete gamma function, $\varphi(\kappa)$ is the digamma function, $\varphi'(\kappa)$ is the trigamma function, P_1 is denoted to the "Log-likelihood Function", P_2 to the " Expected Negative Log-likelihood", P_3 to the "Expected Negative Log-likelihood (

$\eta_j \to \infty$)", and P_4 to the p-th quantile (The result of P_2 and P_3 are derived in Appendix B, see supplementary materials).

Model (wrong /true)	Properties	Mathematical Expression
M_{SEV}/M_{LG}	P_1	$\Im_{SEV} = \sum_{j=1}^{m}\left\{\sum_{i=1}^{n_j}\left\{\delta_i\left(-\log\sigma_e + Z_{ije} - e^{Z_{ije}}\right) - (1-\delta_i)e^{\zeta_{je}}\right\}\right\}$
		$\frac{1}{n}E_{LG}(-\Im_{SEV}) = \sum_{j=1}^{m}\pi_j\left\{\left(1-\Phi_{LG}(\zeta_{jg})\right)e^{\zeta_{je}} + \Phi_{LG}(\zeta_{jg})\left(\log\sigma_e + \frac{\mu_e(x_j)}{\sigma_e}\right) - \int_{-\infty}^{\log\eta_j}\left(\frac{\log t_{ij}}{\sigma_e} - \right.\right.$
	P_2	
		$\frac{1}{n}E_{LG}\left(-\Im_{SEV}\right) = \sum_{j=1}^{m}\pi_j\left\{\log\sigma_e + \frac{\gamma_{0e}+\gamma_{1e}x_j - \mu_g(x_j) - \sigma_g\varphi(\kappa)}{\sigma_e} + exp\left(-\frac{\gamma_{0e}+\gamma_{1e}x_j - \mu_g}{\sigma_e}\right.\right.$
	P_3	
	P_4	$t_p(\boldsymbol{\theta}_e) = e^{\left(\mu_e(x_j)+\sigma_e\Phi_{SEV}^{-1}(p)\right)} = e^{\left[\mu_e(x_j)+\sigma_e\log\left(-\log(1-p)\right)\right]}$
		$\Im_{Nor} = \sum_{j=1}^{m}\left\{\sum_{i=1}^{n_j}\left\{\delta_i\left(-\log\sigma_N - \frac{\log 2\pi}{2} - \frac{Z_{ijN}^2}{2}\right) + (1-\delta_i)\log\left(1-\Phi_{Nor}(\zeta_{jN})\right)\right\}\right\}$
M_{Nor}/M_{L_G}	P_1	
		$\frac{1}{n}E_{LG}(-\Im_{Nor}) = \sum_{j=1}^{m}\pi_j\left\{-(1-\Phi_{LG}(\zeta_{jg}))\log(1-\Phi_{Nor}(\zeta_{jN})) + \Phi_{LG}(\zeta_{jg})\left(\frac{1}{2}\log 2\pi + \log\sigma_N\right.\right.$
		$\left. + \frac{1}{2\sigma_N^2}\int_{-\infty}^{\log\eta_j}\left(\log t_{ij}^2 - 2\log t_{ij}\mu_N(x_j)\right)\phi_{LG}(Z_{ijg})dt_{ij}\right\}$
	P_2	
		$\frac{1}{n}E_{LG}\left(-\Im_{Nor}\right) = \sum_{j=1}^{m}\pi_j\left\{\frac{1}{2}\log 2\pi + \log\sigma_N + \frac{\left(\mu_N(x_j)\mu_g(x_j) - \sigma_g\varphi(\kappa)\right)^2}{2\sigma_N^2} + \frac{\sigma_g^2\varphi'(\kappa)}{2\sigma_N^2}\right\}$
	P_3	
	P_4	$t_p(\boldsymbol{\theta}_N) = e^{\left(\mu_N(x_j)+\sigma_N\Phi_{Nor}^{-1}(p)\right)}$

Model (wrong /true)	Properties	Mathematical Expression
M_{LG}/M_{LG}	P_4	$t_p(\boldsymbol{\theta}_g) = e^{\left(\mu_g(x_j)+\sigma_g \Phi_{LG}^{-1}(p)\right)} = e^{\left(\mu_g(x_j)+\sigma_g \log\left[Q^{-1}(\kappa,p)\right]\right)}$

Table 3 Best parameter setting for mistreated distributions.

Distribution	γ_0^*	γ_1^*	σ^*
SEV	4.42	-0.799	0.48
Normal	4.167	-0.8	0.526

Table 4 Re-partition the sample size between each stress level with test plan 4:2:1.

stress	**sample size**			
	35	**70**	**140**	**280**
Low	20	40	80	160
Medium	10	20	40	80
High	5	10	20	40

Table 5 The simulation result of *RB* (*RV*) for the Weibull and Lognormal under the case of complete data.

Sample size	**Distribution**	**Quantile**				
		0.05	**0.1**	**0.5**	**0.9**	**0.95**
35	**Weibull**	-0.25(4.53)	-0.15(2.22)	0.0397(1.36)	-0.0005(1.17)	-0.034(1.16)
	Lognorm	0.0489(1.	0.0169(1.1	-	0.0258(1.	0.055(1.3

Sample size	Distribution	Quantile				
	al	29)	)	0.0247(1.037)	14)	5)
70	Weibull	-0.249(8.29)	-0.146(3.5)	0.0466(1.61)	0.006(1.25)	-0.028(1.26)
	Lognormal	0.0466(1.43)	0.0151(1.12)	-0.025(1.086)	0.0241(1.18)	0.053(1.52)
140	Weibul	-2.47(14.24)	-0.143(5.47)	0.05(1.94)	0.0094(1.27)	-0.0249(1.32)
	Lognormal	0.0455(1.61)	0.0141(1.128)	-0.026(1.15)	0.0229(1.22)	0.052(1.77)
280	Weibull	-0.246(28.34)	-0.143(10.19)	0.0501(2.46)	0.0095(1.29)	-0.0248(1.43)
	Lognormal	0.045(2.069)	0.0136(1.16)	-0.0267(1.32)	0.0224(1.33)	0.051(2.36)
Theoretical Result	Weibull	-0.25(49.84)	-0.14(17.55)	0.051(3.38)	0.01(1.39)	-0.024(1.77)
	Lognormal	0.044(2.76)	0.013(1.21)	-0.027(1.54)	0.022(1.48)	0.051(3.28)

Table 6 MTTF in days (years) of Weibull and GG_3 distributions for simulated warranty data under different use-rate levels.

Use-rate Level	Distribution	
	Weibull	GG_3
Low	12367 (33)	11190 (30)
Medium	7501 (20)	6787 (18)
High	4549 (12)	4116 (11)